On the three types of Bell's inequalities

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What are the assumptions of Bell's inequalities?

- **Werner: locality + classicality**

  "Bell showed ... that classicality and locality together lead to false empirical conclusions"

- **Maudlin: only locality**

  "the main order of business ought to be demonstrating exactly where the argument presumes classicality"

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  "the main order of business ought to be demonstrating exactly where the argument presumes classicality"
- **Fine, Pitowsky: only classicality**

  "The violation itself has a-priori nothing to do with the principle of locality for it often occurs in cases where spatio-temporal aspects play no role whatever"

Bell's inequalities

$$-1 \leqslant p_{13} + p_{14} + p_{24} - p_{23} - p_{1} - p_{4} \leqslant 0$$

What are quantum probabilities?

1. **Ontological approach:** classical **unconditional** probability

$$p_i = p(A_i)$$

2. **Operational approach:** classical **conditional** probability

$$p_i = p(A_i|a_i)$$

3. **Quantum logical approach:** **quantum** probability

$$p_i = Tr(\rho \hat{A}_i)$$

Bell's inequalities

1. **Classical unconditional probability:**

$$-1 \leqslant p(A_1 \cap B_3) + p(A_1 \cap B_4) + p(A_2 \cap B_3)$$
$$-p(A_2 \cap B_4) - p(A_1) - p(B_3) \leqslant 0$$

2. **Classical conditional probability:**

$$-1 \leqslant p(A_1 \cap B_3|a_1 \cap b_3) + p(A_1 \cap B_4|a_1 \cap b_4)$$
$$+p(A_2 \cap B_3|a_2 \cap b_3) - p(A_2 \cap B_4|a_2 \cap b_4)$$
$$-p(A_1|a_1) - p(B_3|b_3) \leqslant 0$$

3. **Quantum probability:**

$$-1 \leqslant Tr\big(\rho\,(\hat{A}_1\hat{B}_3 + \hat{A}_1\hat{B}_4 + \hat{A}_2\hat{B}_3 - \hat{A}_2\hat{B}_4 - \hat{A}_1 - \hat{B}_3)\big) \leqslant 0$$

Main claims

The violation of the three different types of Bell's inequalities:

1. rules out classical events and also common causes for classical **unconditional** probabilities
2. rules out common causes but not classical events for classical **conditional** probabilities
3. does not rule out common causes for **quantum** probabilities

Main claims

The violation of the three different types of Bell's inequalities:

1. rules out classical events and also common causes for classical **unconditional** probabilities (Fine, Pitowsky)
2. rules out common causes but not classical events for classical **conditional** probabilities (Bell)
3. does not rule out common causes for **quantum** probabilities (Hofer-Szabó and Vecsernyés)

Project

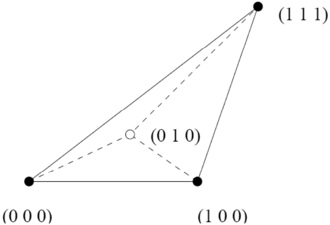
I. Classical events
II. Common causes
III. Relating Bell and Pitowsky
IV. Noncommuting common causes

Pitowsky's question: When can a set of numbers be interpreted as **unconditional** probability of (classical) events and their conjunctions?

When can the numbers p_1, p_2 and p_{12} be the probability of events A_1, A_2 and $A_1 \cap A_2$?

I. Classical events

- Correlation vector: $\vec{p} = (p_1, p_2, p_{12})$
- Vertices: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$
- Classical correlation polytope:



- Facet inequalities:

$$0 \leqslant p_{12} \leqslant p_1 \leq 1$$

$$0 \leqslant p_{12} \leqslant p_2 \leq 1$$

$$p_1 + p_2 - p_{12} \leqslant 1$$

I. Classical events

- Correlation vector: $\vec{p} = (p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24})$
- Facet inequalities:

$$
\begin{aligned}
0 \leqslant p_{ij} \leqslant p_i &\leqslant 1 \\
0 \leqslant p_{ij} \leqslant p_j &\leqslant 1 \qquad i = 1, 2 \;\; j = 3, 4 \\
p_i + p_j - p_{ij} &\leqslant 1 \\
-1 \leqslant p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 &\leqslant 0 \\
-1 \leqslant p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 &\leqslant 0 \\
-1 \leqslant p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 &\leqslant 0 \\
-1 \leqslant p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 &\leqslant 0
\end{aligned}
$$

Pitowsky, 1989: The following statements are equivalent:

1. $\vec{p}$ satisfies Bell's inequalities
2. $\vec{p}$ is in the classical correlation polytope
3. $\vec{p}$ can represent the probability of classical events and their conjunctions

I. Classical events

When can the numbers p_1, p_2 and p_{12} be

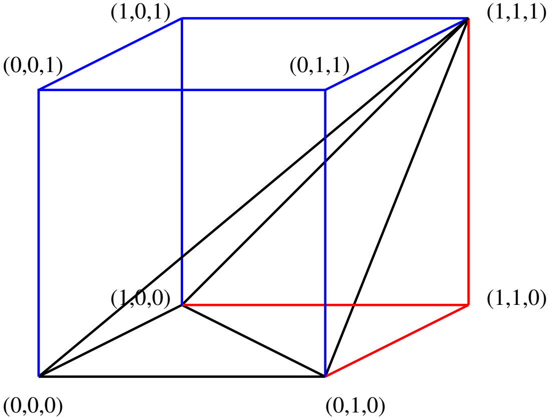
1. **conditional** probabilities:

$$p_1 = p(A_1|a_1), \quad p_2 = p(A_2|a_2), \quad p_{12} = p(A_1 \cap A_2|a_1 \cap a_2)$$

2. quantum probabilities:

$$p_1 = Tr(\rho\,\hat{A}_1), \quad p_2 = Tr(\rho\,\hat{A}_2), \quad p_{12} = Tr(\rho\,\hat{A}_1\hat{A}_2)$$

Polytopes:



II. Common causes

Common Cause Principle: If there is a correlation between two events and there is no direct causal connection between the correlating events, then there always exists a common cause of the correlation

II. Common causes

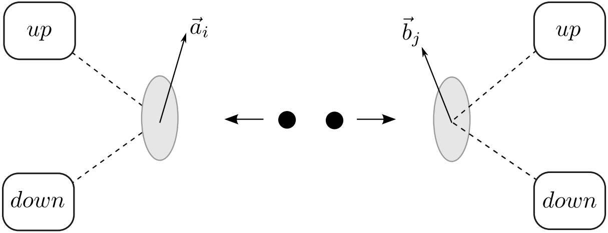
- **Unconditional** correlations:

$$p(A_i \cap B_j) \neq p(A_i)p(B_j)$$

- Joint common cause: partition $\{C_k\}$

$$p(A_i \cap B_j | C_k) \quad = \quad p(A_i|C_k)\, p(B_j|C_k)$$

II. Common causes



- Measurements: a_i, b_j
- Outcomes: A_i, B_j
- **Conditional** correlations:

$$p(A_i \cap B_j | a_i \cap b_j) \neq p(A_i|a_i)p(B_j|b_j)$$

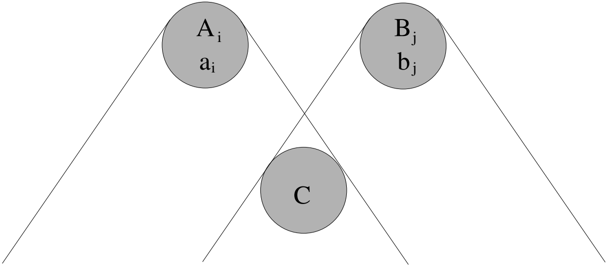
II. Common causes

Bell's question: Is there a common causal explanation of the **conditional** EPR correlations?

II. Common causes

Common causal explanation:

$$
\begin{aligned}
p(A_i \cap B_j | a_i \cap b_j \cap C_k) &= p(A_i | a_i \cap b_j \cap C_k)\, p(B_j | a_i \cap b_j \cap C_k) \\
p(A_i | a_i \cap b_j \cap C_k) &= p(A_i | a_i \cap C_k) \\
p(B_j | a_i \cap b_j \cap C_k) &= p(B_j | b_j \cap C_k) \\
p(a_i \cap b_j \cap C_k) &= p(a_i \cap b_j)\, p(C_k)
\end{aligned}
$$



Common causal explanations imply Bell's **conditional** inequalities:

$$
\begin{aligned}
-1 \leqslant p(A_1 \cap B_3 | a_1 \cap b_3) &+ p(A_1 \cap B_4 | a_1 \cap b_4) \\
+p(A_2 \cap B_3 | a_2 \cap b_3) &- p(A_2 \cap B_4 | a_2 \cap b_4) \\
&-p(A_1 | a_1) - p(B_3 | b_3) \leqslant 0
\end{aligned}
$$

III. Relating Bell and Pitowsky

Pitowsky: Classical events $\Longrightarrow$ Bell's **unconditional** inequalities

Bell: Common causes $\Longrightarrow$ Bell's **conditional** inequalities

Equivalent statements:

1. $\vec{p}$ satisfies Bell's inequalities
2. $\vec{p}$ is in the classical correlation polytope
3. $\vec{p}$ can represent the **unconditional** probability of classical events and their conjunctions
4. ~~$\vec{p}$ can represent the probability of measurement outcomes **conditioned** on measurement choices~~
5. All **conditional** correlations in $\vec{p}$ $p(A_i \cap B_j | a_i \cap b_j) \neq p(A_i | a_i)\, p(B_j | b_j)$ have a common causal explanation
6. All **unconditional** correlations in $\vec{p}$ $p(A_i \cap B_j) \neq p(A_i)\, p(B_j)$ have a joint common cause

III. Relating Bell and Pitowsky

Pitowsky's question: When can a set of numbers be interpreted as **unconditional** probability of (classical) events and their conjunctions?

$$\Updownarrow$$

Does $\vec{p}$ satisfy Bell's inequality?

$$\Updownarrow$$

Bell's question: Is there a common causal explanation of the **conditional** EPR correlations?

- **Quantum probabilities:**

$$-1 \leqslant Tr\big(\rho\,(\hat{A}_1\hat{B}_3 + \hat{A}_1\hat{B}_4 + \hat{A}_2\hat{B}_3 - \hat{A}_2\hat{B}_4 - \hat{A}_1 - \hat{B}_3)\big) \leqslant 0$$

IV. Noncommuting common causes

- **Quantum correlations:**

$$Tr(\rho\,\hat{A}_i\hat{B}_j) \quad \neq \quad Tr(\rho\,\hat{A}_i)\,Tr(\rho\,\hat{B}_j)$$

- **Quantum common cause:** orthogonal projections $\{C_k\}$

$$\frac{Tr(\rho\,\hat{C}_k\hat{A}_i\hat{B}_j\hat{C}_k)}{Tr(\rho\,\hat{C}_k)} \quad = \quad \frac{Tr(\rho\,\hat{C}_k\hat{A}_i\hat{C}_k)}{Tr(\rho\,\hat{C}_k)}\,\frac{Tr(\rho\,\hat{C}_k\hat{B}_j\hat{C}_k)}{Tr(\rho\,\hat{C}_k)}$$

IV. Noncommuting common causes

The violation of the quantum Bell inequalities does not exclude **noncommuting common causes**!

IV. Noncommuting common causes

$\frac{1}{2}\big(\hat{A}_1(\hat{B}_3 + \hat{B}_4) + \hat{A}_2(\hat{B}_3 - \hat{B}_4)\big)$ is a **witness operator** testing the separability/entanglement of quantum states

Conclusions

1. **Classical unconditional probability:**

$$-1 \leqslant p(A_1 \cap B_3) + p(A_1 \cap B_4) + p(A_2 \cap B_3) - p(A_2 \cap B_4) - p(A_1) - p(B_3) \leqslant 0$$

2. **Classical conditional probability:**

$$-1 \leqslant p(A_1 \cap B_3 | a_1 \cap b_3) + p(A_1 \cap B_4 | a_1 \cap b_4) + p(A_2 \cap B_3 | a_2 \cap b_3) - p(A_2 \cap B_4 | a_2 \cap b_4) - p(A_1 | a_1) - p(B_3 | b_3) \leqslant 0$$

3. **Quantum probability:**

$$-1 \leqslant Tr\big(\rho\,(\hat{A}_1\hat{B}_3 + \hat{A}_1\hat{B}_4 + \hat{A}_2\hat{B}_3 - \hat{A}_2\hat{B}_4 - \hat{A}_1 - \hat{B}_3)\big) \leqslant 0$$

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IV. Separable states

- $\mathcal{A}$, $\mathcal{B}$: two mutually commuting C^* **subalgebras** of the C^* algebras $\mathcal{C}$
- **Bell operator:** an element $\hat{R}$ of the set

$$\left\{ \frac{1}{2}\left[\hat{A}_1(\hat{B}_3 + \hat{B}_4) + \hat{A}_2(\hat{B}_3 - \hat{B}_4)\right] \right\}$$

  where $\hat{A}_i = \hat{A}_i^* \in \mathcal{A}$, $\hat{B}_i = \hat{B}_i^* \in \mathcal{A}$ and $-\mathbf{1} \leqslant \hat{A}_i, \hat{B}_i \leqslant \mathbf{1}$

- For **separable** ρ : $\quad |Tr(\rho \hat{R})| \leqslant 1$
- $\hat{R}$ is a **witness operator** testing the separability of states

Conclusion

To what question is Bell's inequality an answer?

I. Question: When can numbers represent probabilities?

II. Question: When do correlations have a common causal explanation?

III. Question: ???