An attempt to understand the PBR no-go theorem

Gábor Hofer-Szabó

Claim of PBR:

- "The quantum state cannot be interpreted statistically" (title of the PBR paper)

What is a statistical interpretation?

- **Statistical interpretation:**
  - M: measurement
  - $\{A_i\}$: measurement outcomes ($i = 1 \ldots n$)
  - p: probability measure on the algebra generated by $\{A_i\}$
  - The pair $(\{A_i\}, p)$ has a *statistical interpretation* or *model* if there exists a (Λ, μ, M_i) such that for any outcome A_i:

    $$p(A_i) = \int_\Lambda M_i(\lambda) d\mu(\lambda)$$

    The response function $M_i(\lambda)$ represents the probability of getting the outcome A_i if the value of the hidden variable is λ. Obviously, $\sum_i M_i(\lambda) = 1$ for any $\lambda \in \Lambda$.
  - (The statistical model is *deterministic*, if $M_i(\lambda) \in \{0, 1\}$ for any i and λ.)

- **An example:**
  - From a hat containing many coins we draw one coin then toss it and register the result. Repeating the tossings we end up with the probability measure:

    $$\begin{aligned} p(Head) &= 0.8 \\ p(Tail) &= 0.2 \end{aligned}$$

  - Three statistical models:
    * $\Lambda = [0, 1]$
    * Response functions ('bias parameters'):

      $$\begin{aligned} M_H(\lambda) &= \lambda \\ M_T(\lambda) &= 1 - \lambda \end{aligned}$$

    * Three probability measures:

      $$\begin{aligned} \mu_1 &= 0.2\delta(0) + 0.8\delta(1) \\ \mu_2 &= \delta(0.8) \\ \mu_3 &= \begin{cases} \frac{2}{d} & \text{if } \lambda \in [0.8 - d, 0.8 + d] \\ 0 & \text{othervise} \end{cases} \end{aligned}$$

- Interpretation of the three models:
    - (Λ, μ_1, M_i): property interpretation (deterministic)
    - (Λ, μ_2, M_i): propensity interpretation
    - (Λ, μ_3, M_i): mixed interpretation

- **Statistical interpretation of QM:**
  - M: (projective) measurement
  - $\{P_i^M\}$: measurement outcomes (an orthonormal basis)
  - Probability measure: $Tr(W \cdot\,)$
  - Statistical interpretation: there is a (Λ, μ, M_i) such that

$$Tr(WP_i^M) = \int_\Lambda M_i(\lambda) d\mu(\lambda)$$

- **Outcome space $\{A_i\}$ with a two states: p, p'**
  - $(\{A_i\}, p, p')$ has a *statistical model* if there exists a $(\Lambda, \mu, \mu', M_i)$ such that:

$$\begin{aligned} p(A_i) &= \int_\Lambda M_i(\lambda) d\mu(\lambda) \\ p'(A_i) &= \int_\Lambda M_i(\lambda) d\mu'(\lambda) \end{aligned}$$

  Again, $\sum_i M_i(\lambda) = 1$ for any $\lambda \in \Lambda$.

- **The crucial distinction of PBR:** a statistical model of $(\{A_i\}, p, p')$ is
  - **ontic**: if $Supp(\mu) \cap Supp(\mu') = \emptyset$
  - **epistemic**: if $Supp(\mu) \cap Supp(\mu') \neq \emptyset$

- **Or more precisely:** a statistical model of $(\{A_i\}, p, p')$ is
  - **ontic**: if $\mu\big(Supp(\mu) \cap Supp(\mu')\big) = \mu'\big(Supp(\mu) \cap Supp(\mu')\big) = 0$
  - **epistemic**: if $\mu\big(Supp(\mu) \cap Supp(\mu')\big) \neq 0$ or $\mu'\big(Supp(\mu) \cap Supp(\mu')\big) \neq 0$.

- **Claim of PBR:** "The quantum state cannot be interpreted **statistically**" = "The quantum state cannot be interpreted **epistemically**"

The argument of PBR

- **Claim:** If $\mathcal{A}$ is the operator algebra on a finite Hilbert space and ψ and ψ' are two pure states then there exists *no epistemic* statistical model of ψ and ψ' which is compatible with the Born rule.

- **Proof:** Suppose that ψ and ψ' has a statistical interpretation that is for every self adjoint $M \in \mathcal{A}$ with $i = 1 \ldots n$ eigenvalues there exists a $(\Lambda, \mu, \mu', M_i)$ such that

$$\begin{aligned} Tr(W_\psi\, P_i^M) &= \int_\Lambda M_i(\lambda) d\mu(\lambda) \\ Tr(W_{\psi'}\, P_i^M) &= \int_\Lambda M_i(\lambda) d\mu'(\lambda) \end{aligned}$$

  where P_i^M is the projection onto the ith eigensubspace of M.

- **Trick:** to find a measurement M (that is an orthogonal set of projections) **separating** the two states:

$$\begin{aligned} Tr(W_\psi \, P_1^M) &= 0 \\ Tr(W_{\psi'} \, P_2^M) &= 0 \end{aligned}$$

- It follows:

$$\begin{aligned} \int_\Lambda M_1(\lambda) d\mu(\lambda) = 0 \quad &\Longrightarrow \quad M_1(\lambda) = 0 \;\text{ if } \lambda \in Supp(\mu) \\ \int_\Lambda M_2(\lambda) d\mu'(\lambda) = 0 \quad &\Longrightarrow \quad M_2(\lambda) = 0 \;\text{ if } \lambda \in Supp(\mu') \end{aligned}$$

(at least, if $\mu(\{\lambda\}) \neq 0$ and $\mu'(\{\lambda\}) \neq 0$)

- **Epistemic model:** There is a λ^* in $Supp(\mu) \cap Supp(\mu') \quad \Longrightarrow \quad M_1(\lambda^*) = M_2(\lambda^*) = 0$
- **But:** from the normalization: $M_1(\lambda) + M_2(\lambda) = 1$ for any $\lambda \in \Lambda$, contradiction!

The PBR theorem

- **Step 1.** If $\psi \perp \psi'$: there is a separating mearurement: $P_1^M = P_\psi$ and $P_2^M = P_{\psi'}$
- **Step 2.** If $\psi \not\perp \psi'$: no such separating measurement M
  - **Trick:** Consider multiple copies of the system
  - Consider a system S with two pure states $\psi = |0\rangle$ and $\psi' = |+\rangle$ (where $\{|0\rangle, |1\rangle\}$ is a basis in H_2 and $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$).
  - Now, consider two uncorrelated copies of S. The system $S \times S$ can have the following four pure tensor states composed of $|0\rangle$ and $|+\rangle$:

$$\begin{aligned} &|0\rangle \otimes |0\rangle \\ &|+\rangle \otimes |0\rangle \\ &|0\rangle \otimes |+\rangle \\ &|+\rangle \otimes |+\rangle \end{aligned}$$

  - PBR show that there exists four orthogonal projections in $H_2 \otimes H_2$, namely

$$\begin{aligned} |P_1^M\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \\ |P_2^M\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle) \\ |P_3^M\rangle &= \frac{1}{\sqrt{2}}(|+\rangle \otimes |1\rangle + |-\rangle \otimes |0\rangle) \\ |P_4^M\rangle &= \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) \end{aligned}$$

    representing a joint measurement on $S \times S$ such that

$$\begin{aligned} |P_1^M\rangle &\perp |0\rangle \otimes |0\rangle \\ |P_2^M\rangle &\perp |+\rangle \otimes |0\rangle \\ |P_3^M\rangle &\perp |0\rangle \otimes |+\rangle \\ |P_4^M\rangle &\perp |+\rangle \otimes |+\rangle \end{aligned}$$

– Therefore

$$
\begin{aligned}
Tr(W_{|0\rangle\otimes|0\rangle}\, P_1^M) &= 0\\
Tr(W_{|+\rangle\otimes|0\rangle}\, P_2^M) &= 0\\
Tr(W_{|0\rangle\otimes|+\rangle}\, P_3^M) &= 0\\
Tr(W_{|+\rangle\otimes|+\rangle}\, P_4^M) &= 0
\end{aligned}
$$

– Suppose that $\{|0\rangle, |+\rangle\}$ has a **statistical model**: $(\Lambda, \mu_0, \mu_+, M_i')$

– **Independence assumption:** Since there is no correlation between the two copies, the statistical model of

$$\Big\{|0\rangle \otimes |0\rangle, |+\rangle \otimes |0\rangle, |0\rangle \otimes |+\rangle, |+\rangle \otimes |+\rangle\Big\}$$

is of the form:

$$\Big(\Lambda \times \Lambda, \mu_0\mu_0,\ \mu_+\mu_0,\ \mu_0\mu_+,\ \mu_+\mu_+, M_i(\lambda, \lambda')\Big)$$

– Therefore

$$\int_{\Lambda\times\Lambda} M_1(\lambda,\lambda')d\mu_0(\lambda)\, d\mu_0(\lambda') = 0 \implies M_1(\lambda,\lambda') = 0 \text{ if } (\lambda,\lambda') \in Supp(\mu_0) \times Supp(\mu_0)$$

$$\int_{\Lambda\times\Lambda} M_2(\lambda,\lambda')d\mu_+(\lambda)\, d\mu_0(\lambda') = 0 \implies M_2(\lambda,\lambda') = 0 \text{ if } (\lambda,\lambda') \in Supp(\mu_+) \times Supp(\mu_0)$$

$$\int_{\Lambda\times\Lambda} M_3(\lambda,\lambda')d\mu_0(\lambda)\, d\mu_+(\lambda') = 0 \implies M_3(\lambda,\lambda') = 0 \text{ if } (\lambda,\lambda') \in Supp(\mu_0) \times Supp(\mu_+)$$

$$\int_{\Lambda\times\Lambda} M_4(\lambda,\lambda')d\mu_+(\lambda)\, d\mu_+(\lambda') = 0 \implies M_4(\lambda,\lambda') = 0 \text{ if } (\lambda,\lambda') \in Supp(\mu_+) \times Supp(\mu_+)$$

– **Epistemic model:** There is a λ^* in $Supp(\mu_0) \cap Supp(\mu_+) \implies M_i(\lambda^*, \lambda^*) = 0$

– **But:** from the normalization: $\sum_i M_i(\lambda, \lambda') = 1$ for any $(\lambda, \lambda') \in \Lambda \times \Lambda$, contradiction!

- **Step 3.** Let $|\psi\rangle$ and $|\psi'\rangle \in H_2$ be *any* two states in H_2. Now, consider the 2^n pure states in $\otimes_n H_2$ (where the choice of n depends on the angle between $|\psi\rangle$ and $|\psi'\rangle$):

$$
\begin{aligned}
&|\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \otimes |\psi\rangle\\
&|\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \otimes |\psi'\rangle\\
&.\\
&.\\
&.\\
&|\psi'\rangle \otimes |\psi'\rangle \otimes \cdots \otimes |\psi'\rangle \otimes |\psi'\rangle
\end{aligned}
$$

Then there exis 2^n orthogonal projections

$$
\begin{aligned}
&|P_1^M\rangle\\
&|P_2^M\rangle\\
&.\\
&.\\
&.\\
&|P_{2^n}^M\rangle
\end{aligned}
$$

in $\otimes_n H_2$ such that

$$\begin{array}{rcl} |P_1^M\rangle & \perp & |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \otimes |\psi\rangle \\ |P_2^M\rangle & \perp & |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \otimes |\psi'\rangle \\ & . & \\ & . & \\ & . & \\ |P_{2^n}^M\rangle & \perp & |\psi'\rangle \otimes |\psi'\rangle \otimes \cdots \otimes |\psi'\rangle \otimes |\psi'\rangle \end{array}$$

Ontic *vs.* epistemic statistical interpretations

- **Standard**: A statistical model $(\Lambda, \mu, \mu', M_i)$ of $\left(\{P_i^M\}, \psi, \psi'\right)$ is
  - **ontic**: if both μ and μ' are (pure, dispersion-free) Dirac measures
  - **epistemic**: if either μ or μ' is *not* a Dirac measure (mixed, dispersed)
- **PBR**: A statistical model $(\Lambda, \mu, \mu', M_i)$ of $\left(\{P_i^M\}, \psi, \psi'\right)$ is
  - **ontic**: if $Supp(\mu) \cap Supp(\mu') = \emptyset$
  - **epistemic**: if $Supp(\mu) \cap Supp(\mu') \neq \emptyset$
- **Why?**
- **History:**
  - Spekkens, 2008: Defending the epistemic view of the quantum states in toy models
  - Spekkens, 2010:
    * Einstein's 1935 argument ($\neq$ EPR paper) is an argument against *ontic* models of the quantum state: assuming *local causality* there can be many quantum states associated with the same λ.

      > "Now what is essential is exclusively that ψ_B and $\psi_{\underline{B}}$ are in general different from one another. I assert that this difference is incompatible with the hypothesis that the description is correlated one-to-one with the physical reality (the real state). After the collision, the real state of (AB) consists precisely of the real state of A and the real state of B, which two states have nothing to do with one another. The real state of B thus cannot depend upon the kind of measurement I carry out on A. But then for the same state of B there are two (in general arbitrarily many) equally justified ψ_B, which contradicts the hypothesis of a one-to-one or complete description of the real states." (Einstein's letter to Schrödinger, 1935)
    * The Bell theorems using again *causal locality* exclude both *ontic* and *epistemic* models of the quantum
    * But without assuming *local causality* "it remains unclear to what extent a ψ-epistemic ontological model of quantum theory is even possible." (Spekkens, 2010, 152.)
  - PBR, 2012: No *epistemic* interpretation of the quantum state

- **Two quotes:**
  - "The reader might well be wondering why we do not admit that any ψ-incomplete model is 'epistemic', simply because it associates a probability distribution of nontrivial width over Λ with each quantum state. We admit that although it might be apt to say that ψ-incomplete models have an epistemic character, the question of interest here is whether *pure quantum states* have an epistemic character. It is for this reason that we speak of whether a model is 'ψ-epistemic' rather than simply 'epistemic'. By our definitions, ψ has an ontic character if and only if a variation of ψ implies a variation of reality and an epistemic character if and only if a variation of ψ does not necessarily imply a variation of reality." (Harrigan, Spekkens, 2010, 132.)
  - "The most important property of the ontic state ... [is that]: it sceens off the preparation from the measurement. More precisely, for all measurement, the variable that runs over the outcomes of the measurement and the variable that runs over the preparation procedures are conditionally independent given the ontic state." (Harrigan, Spekkens, 2010, 134.)
- **My (desperate) attempts:**
  - In ontic models quantum states *supervene* on hidden variable states.
  - In ontic models quantum states *cluster* the hidden variable space.
  - In ontic models "the connection between hidden variables and states is *functional*." (Shlosshauer, Fine, 2012, 3.)
  - In ontic models quantum states are *among* the hidden variables.
  - Ontic models are *non-redundant* in the sense that different states are modeled on different regions of the hidden variable space.
  - Using a Bayesian framework one might say that if p and p' are ontic states then the change from p to p' *cannot be a change simply in one's knowledge* since there is no underlying $\lambda \in \Lambda$ which would make both p and p' true. Something had to happen in the world which justifies this change.
- **Schlosshauer, Fine, 2012:** Ontic and epistemic models can be transformed into one another.

  > "The theorem shows the price we may have to pay for a hidden-variables model that is not segregated [ontic]. We put it this way to make clear that PBR do not show that mixed [epistemic] models are predictively flawed or fail to yield the correct quantum statistics for some observables or states of a given system. Rather, PBR demonstrate a possible difficulty for hidden-variables models in forming composites of identically prepared systems." (3.)

References

Schlosshauer M., A. Fine (2012). "Implications of the Pusey–Barrett–Rudolph no-go theorem," URL = http://arxiv.org/pdf/1203.4779.

Harrigan, N, R. W. Spekkens (2010). "Einstein, Incompleteness, and the Epistemic View of Quantum States," *Foundations of Physics*, **40**, 125-157.

Pusey, M. F., J. Barrett, T. Rudolph (2012). "The quantum state cannot be interpreted statistically," URL = arxiv.org/abs/1111.3328.

Spekkens, R. W. (2008). "In the defense of the epistemic view of quantum states: toy theory," http://arxiv.org/abs/quant-ph/0401052.

Miscellaneous

Statistical interpretation of PBR

- **PBR:** $(\{A_i\}, p, p')$ has a *statistical model* if there exists a $(\Lambda, \mu, \mu', \mathbf{M_i})$ such that:

$$\begin{aligned} p(A_i) &= \int_\Lambda M_i(\lambda) d\mu(\lambda) \\ p'(A_i) &= \int_\Lambda M_i(\lambda) d\mu'(\lambda) \end{aligned}$$

- **Standard:** $(\{A_i\}, p, p')$ has a *statistical model* if there exists a $(\Lambda, \mu, \mu', \mathbf{M_i}, \mathbf{M_i'})$ such that:

$$\begin{aligned} p(A_i) &= \int_\Lambda M_i(\lambda) d\mu(\lambda) \\ p'(A_i) &= \int_\Lambda M_i'(\lambda) d\mu'(\lambda) \end{aligned}$$

  (Or more liberally, $\mathbf{\Lambda} \neq \mathbf{\Lambda'}$.)

- If $M_i \neq M_i'$, then there is no contradiction.
- Why should we require that the response functions are independent of the state? One might prepare the coins in the hat by magnetic field in the one case and by biased mass distibution in the other.

Quantum Bayesianism

- **Quantum Bayesianism** (Caves, Fuchs, Schack, Zeilinger, Bub): quantum physics is only about information
- "Basically, PBR call something 'statistical' if two people, who live in the same universe but have different information, could rationally disagree about it. . . As for what 'rational' means, all we'll need to know is that a rational person can never assign a probability of 0 to something that will actually happen." (Scott Aaronson)
  - If I think that the next throw of a dice will be even and it turns out to be 6 $\longrightarrow$ I am *rational*
  - If I think that the next throw of a dice will be odd and it turns out to be 6 $\longrightarrow$ I am *irrational*