Quantum theory and local causality

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Syllabus

Contents

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1 Real world quantum correlations

Abstract: Real world experiments will be presented producing quantum correlations which call for a causal explanation.

Literature: Redhead, 1987, Sec. 4.5; Scarani 2006, Ch. 3

- **Introduction.** The foundational researches of quantum mechanics have been launched mostly by two seminal papers: the Einstein-Podolsky-Rosen (1935) paper

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

and John Bell's 1964 paper.

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ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

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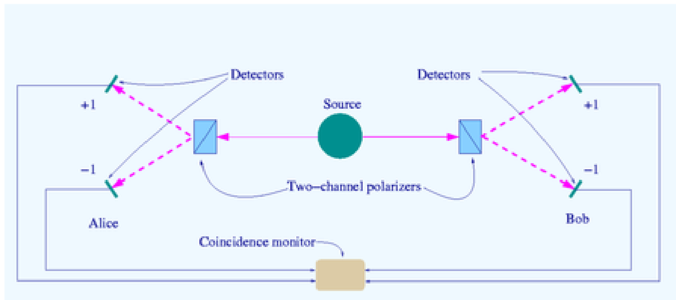
(*Received* 4 *November* 1964)

I. Introduction

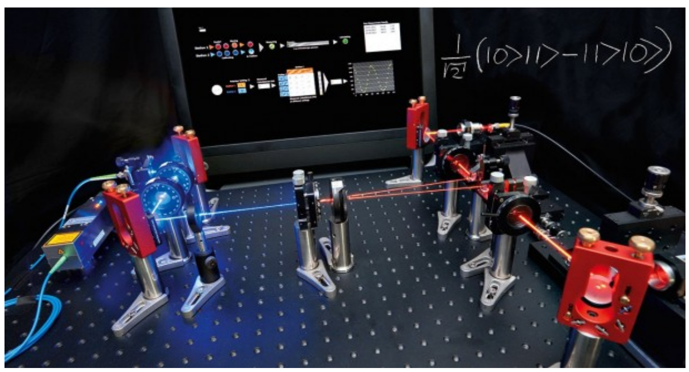
THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

These papers triggered an intense philosophical research on such concepts as locality, causality, probability, realism, etc in relation to QM. All the discussions circled around certain spooky correlations. Next, we delineate one such real world experiment.

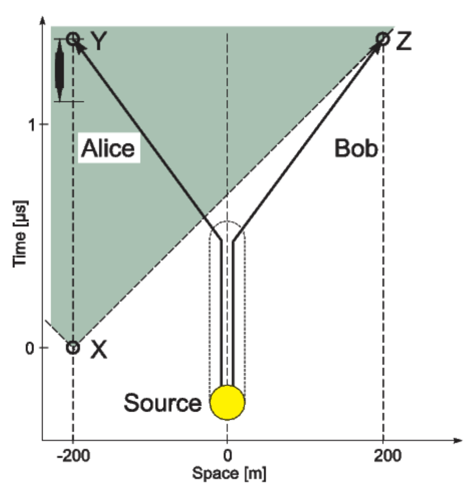
- **The Weihs et al. (1998) experiment.** In this experiment observers were spatially separated by 400 m across the Innsbruck university campus. They used polar-



ization entangled[1] photon pairs transmitted to the observers through optical fibers. The difference in fiber length was less that 1 m hence the photons were registered



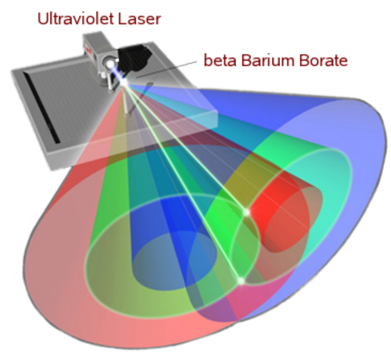
simultaneously within 5 ns. Each individual measurement on both side consisted in choosing the direction, setting the analyzer and registering the particle. Using high



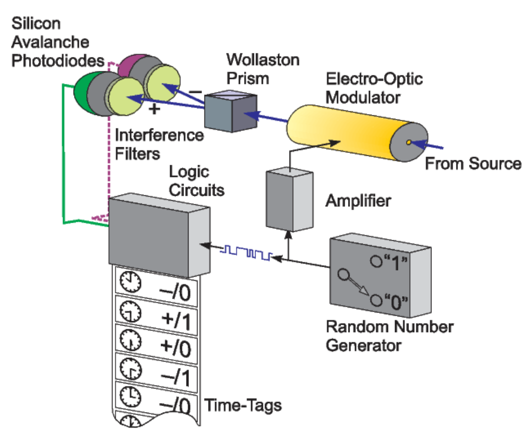
[1] $\Psi = \frac{1}{\sqrt{2}}\big(|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2\big)$

speed physical[2] random number generators and fast electro-optic modulators this process lasted 100 ns, much shorter than the 1,3 μs time needed for communication between the observers.

Technological details. The particle source was degenerate type-II[3] parametric down-conversion where the BBO (barium borate) nonlinear crystal was pumped with light of 351 nm from an Argon-ion-laser providing polarization entangled photon pairs of 702 nm.



Each observers switched the polarization of the beam between 0° and 45° with an electro-optic modulator. The orientation of the modulator was determined by a physical random number generator with 10 ns mean interval. Modulated photons were transmitted to a beam splitter (Wollaston prism) sending the photons of different polarization into two photomultipliers (silicon avalanche photodiodes) detecting the photons. Detectors counted 10-15.000 photons per second with dark count rate[4] of a few hundreds per second. Events where both detectors (on the same side) register a photon within a 2 ns time window are ignored.



[2]Not numerical pseudo-random generators, since their state can be predetermined. In the former Aspect (1982) experiment periodic sinusoidal switching has been used.

[3]Photons have perpendicular polarizations.

[4]When thermally-generated carriers fire the avalanche process.

The time scale of the two observers were synchronized by laser pulses through a second optical fiber up to 20 ns accuracy. Time tags of outcomes are registered independently and compared only after the measurement. In a typical observation 14.700 coincidence events have been measured. QM has been confirmed with 30 standard deviations.

- **Other notable Bell tests:**

  1972: Freedman and Clauser at Berkeley perform the first Bell tests. They measure the pairs separately.

  1982: Aspect, Dalibard and Roger experiment at Paris-Sud University at Orsay. The interferometer contains only one photon per experiment; they modify the interferometers during the flight of the particles (using periodic sinusoidal switching).

  Anton Zeilinger group:

  - **1997:** quantum teleportation
  - **1998: Weihs et al.** experiment in Innsbruck. First time the locality loophole is closed by switching the interferometers by physical random generator.
  - **1999:** interference with fullerenes (C_{60}, 1080 particles) $\rightarrow$ insulin, biological molecules?
  - **2004:** quantum teleportation under the Danube

  Nicolas Gisin group:

  - **1996:** quantum cryptography over 20 km distance.
  - **1998:** Bell experiment in Geneva. The distance between the two analysis stations, Bellevue and Bernex is 11 km. They used the Swiss Telecom.

  2010: Scheidl et al. conducted an entangled photon experiment between the islands of La Palma and Tenerife separated by a distance of 144 km (33.000 counts per second).

  2013: Ursin proposed to place the detectors on satellites (for example on the International Space Station) orbiting at an altitude of 500 km.

  Fermion experiments. The Rauch group performed neutron interferometry experiments at the Laue-Langevin Institute in Grenoble; the Sakai group performed proton experiments at RIKEN Japan. In what follows such a fermion experiment will be analyzed.

- **Main loopholes:**

1. **Locality loophole**: the measurement choices may influence the measurement outcome in the opposite wing (even if the freedom-of-choice loophole is avoided):

$$p(A_i|a_i \wedge b_j \wedge C_k) \neq p(A_i|a_i \wedge C_k) \tag{1}$$

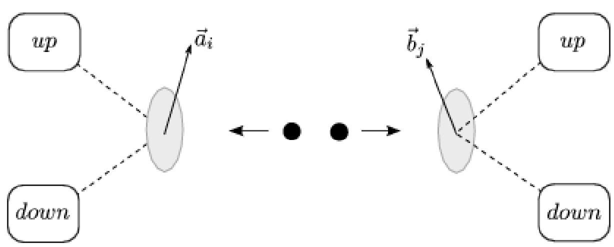
(see the notation in Class 4.) The locality loophole can be avoided by quick and random switching between the measurement choices during the flight of the photons, first achieved in the Weihs (1998) experiment.

2. **Freedom-of-choice loophole:** hidden communication between the source and the measurement choices:

$$p(a_i \wedge b_j \wedge C_k) = p(a_i \wedge b_j)\, p(C_k) \tag{2}$$

3. **Fair-sampling or detection loophole:** an inefficient detection scheme might yield measured photon statistics that differ substantially from the actual photon statistics. (See "police radar example" of Scarani, p. 86.)

- **The EPR-Bohm scenario.** Consider the Bohm version of the EPR experiment with a pair of spin-$\frac{1}{2}$ particles prepared in the singlet state. Let a_i denote the event



that the measurement apparatus is set to measure the spin in direction $\mathbf{a}_i$ in the left wing where i is an element of an index set I of spatial directions perpendicular to the beam; and let $p(a_i)$ stand for the probability of a_i. Let b_j and $p(b_j)$ respectively denote the same for direction $\mathbf{b}_j$ in the right wing where j is again in the index set I. (Note that $i = j$ does not mean that $\mathbf{a}_i$ and $\mathbf{b}_j$ are parallel directions.) Furthermore, let $p(A_i)$ stand for the probability that the spin measurement in direction $\mathbf{a}_i$ in the left wing yields the result "up" and let $p(B_j)$ be defined in a similar way in the right wing for direction $\mathbf{b}_j$.

Now, experiments yield the following conditional probabilities:

$$p(A_i|a_i) = \frac{1}{2} \tag{3}$$

$$p(B_j|b_j) = \frac{1}{2} \tag{4}$$

$$p(A_i \wedge B_j|a_i \wedge b_j) = \frac{1}{2}\sin^2\left(\frac{\theta_{a_i b_j}}{2}\right) \tag{5}$$

where $\theta_{a_i b_j}$ denotes the angle between directions $\mathbf{a}_i$ and $\mathbf{b}_j$.

Thus, there is a (conditional) correlation for any non-perpendicular directions $\mathbf{a}_i$ and $\mathbf{b}_j$:

$$p(A_i \wedge B_j | a_i \wedge b_j) \neq p(A_i | a_i) p(B_j | b_j) \tag{6}$$

Specially, if the measurement directions $\mathbf{a}_i$ and $\mathbf{b}_j$ are parallel, then there is a perfect anticorrelation between the outcomes A_i and B_j:

$$p(A_i \wedge B_j | a_i \wedge b_j) = 0 \tag{7}$$

A further empirical fact is the so-called surface locality that is for any $i, j \in I$ the following relations hold

$$p(A_i | a_i) = p(A_i | a_i \wedge b_j) \tag{8}$$

$$p(B_j | b_j) = p(B_j | a_i \wedge b_j) \tag{9}$$

Question: What is the causal explanation of these correlations?

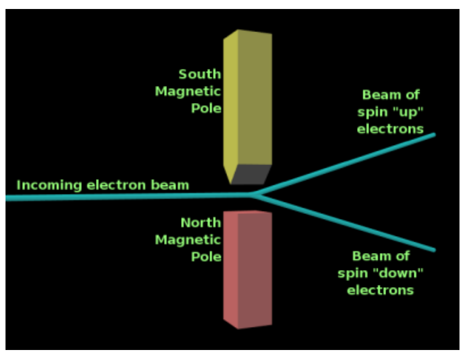
2 The quantum theory of spin

Abstract: The mathematical formalism of spin will be presented.

Literature: Redhead, 1987, Sec. 1.7-8; Ballentine, 1998, Sec. 7.4)

- **Minimal interpretation of QM:** A physical system is represented by a Hilbert space $\mathcal{H}$. Any state of the system is represented by a density operator W, and any observable $\mathcal{A}$ is represented by a self-adjoint operator A acting on $\mathcal{H}$. The evolution of the state is governed by the Schrödinger equation. The probability $p^W(a_i|\mathcal{A})$ of obtaining an outcome a_i provided the observable $\mathcal{A}$ is measured on a system in state W is given by the formula $\mathrm{Tr}(WP_i^A)$ where P_i^A is the projection operator on the eigenvalue a_i of A (Born rule).

- **Spin** is an internal angular momentum of elementary particles. The concept was first proposed and later work out by Pauli. It has been experimentally verified in the Stern-Gerlach experiment in 1922: a beam of electrons is sent through an



  inhomogeneous magnetic field. Since the force on a particle with magnetic moment $\boldsymbol{\mu}$ in an inhomogeneous magnetic field $\mathbf{B}$ is $\boldsymbol{\nabla}(\boldsymbol{\mu}\mathbf{B})$, one expects that—because the angle between $\boldsymbol{\mu}$ and $\mathbf{B}$ differs for each electron—electrons will be deflected in the field continuously. Instead, they are deflected either up or down by a specific amount irrespectively of the orientation of the field.

- **The quantum theory of spin**. The value of the spin of an electron can be $\pm\frac{1}{2}\hbar$ ($\hbar := \frac{h}{2\pi}$), but usually the natural unit $\frac{1}{2}\hbar = 1$ is used. The spin of an electron is represented in $\mathcal{H} = \mathbb{C}^2$ (or more precisely in $\mathbb{C}P^1 := P(\mathbb{C}^2)$) with the (computational) basis:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **Observables** (self-adjoint operators) are (real) linear combination of the Pauli matrices (represented in the above basis) and the identity

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some relations ($i, j, k \in \{x, y, z\}$):

$$\begin{aligned}
\sigma_i^2 &= \mathbf{1} \\
[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i &= 2i\,\varepsilon_{ijk}\sigma_k \\
\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i &= 2i\,\delta_{ij}\mathbf{1} \\
\mathrm{Tr}\,\sigma_i &= 0 \\
\mathrm{Det}\,\sigma_i &= -1
\end{aligned}$$

- **States.** Pure states are represented by rays in $\mathbb{C}P^1$. Any unit vector of $\mathbb{C}^2$ is of the form:

$$e^{i\gamma}\big(\cos(\theta/2)\,|1\rangle + e^{i\varphi}\sin(\theta/2)\,|0\rangle\,\big) = e^{i\gamma}\begin{pmatrix}\cos(\theta/2)\\ e^{i\varphi}\sin(\theta/2)\end{pmatrix}$$

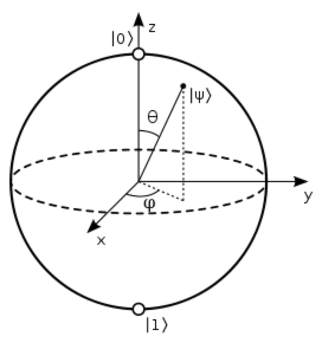
with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ and hence any ray in $\mathbb{C}P^1$ is of the form:

$$|\psi\rangle = \big(\cos(\theta/2)\,|1\rangle + e^{i\varphi}\sin(\theta/2)\,|0\rangle\,\big) = \begin{pmatrix}\cos(\theta/2)\\ e^{i\varphi}\sin(\theta/2)\end{pmatrix}$$

giving rise to the so-called Bloch sphere representation of states:

$$|\psi\rangle_{(\theta,\varphi)} \to \mathbf{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

where $\mathbf{r}$ is unit vector in $\mathbb{R}^3$. The representation is unique except for $|1\rangle$ and $|0\rangle$.



Moreover, $\mathbf{r}$ and $|\psi\rangle$ rotate "nicely":

$$\begin{array}{ccc}
\mathbf{r} & \xrightarrow{O_{\mathbf{n}}(\theta)=e^{i\theta\mathbf{n}\boldsymbol{\tau}}\;\in\; SO(3)} & \mathbf{r'} \\
\downarrow & & \downarrow \\
|\psi\rangle_{\mathbf{r}} & \xrightarrow{U_{\mathbf{n}}(\theta)=e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}}\;\in\; SU(2)} & |\psi'\rangle_{\mathbf{r'}}
\end{array}$$

The corresponding projections (projecting on the ray $|\psi\rangle_{\mathbf{r}}$) are:

$$P_{\mathbf{r}} = \frac{1}{2}(\mathbf{1} + \mathbf{r}\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix}1+\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & 1-\cos\theta\end{pmatrix}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Mixed states are convex combination of pure states represented density operators

$$W = \frac{1}{2}(\mathbf{1} + \mathbf{r}\boldsymbol{\sigma})$$

with $|\mathbf{r}| \leqslant \mathbf{1}$.

- **Eigenvalues.** $|\psi\rangle_{\mathbf{r}}$ is the eigenvector of the spin observable

$$\mathbf{r}\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

with eigenvalue $+1$. In QM the spin measurement in direction $\mathbf{r}$ is represented by solving the $\mathbf{r}\boldsymbol{\sigma}|\psi_{\mathbf{r}}\rangle = s|\psi_{\mathbf{r}}\rangle$ eigenvalue problem:

  - Eigenvalue: $\begin{vmatrix} \cos\theta - s & e^{-i\varphi}\sin\theta \\ e^{-i\varphi}\sin\theta & -\cos\theta - s \end{vmatrix} = -(\cos^2\theta - s^2) - \sin^2\theta = s^2 - 1 = 0 \Longrightarrow s = \pm 1$
  - Eigenstates: unnormalized: $\begin{pmatrix} \pm e^{-i\varphi}\sin\theta \\ 1 \pm \cos\theta \end{pmatrix}$
    normalized (by trigonometry): $|\psi_{\mathbf{r}+}\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix}$, $|\psi_{\mathbf{r}-}\rangle = \begin{pmatrix} -e^{-i\varphi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$
  - Events: projections $P_{\mathbf{r}}$: "the spin of the system in direction $\mathbf{r}$ is $+1$"
  - Special directions: solving the $\sigma_k|\psi_{ki}\rangle = a_{ki}|\psi_{ki}\rangle$ $(k = x, y, z)$ eigenvalue problems:
    * Solution: $\sigma_k|\psi_{k\pm}\rangle = \pm 1|\psi_{k\pm}\rangle$ where the eigenstates are:

$$|\psi_{x+}\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |\psi_{y+}\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad |1\rangle = |\psi_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|\psi_{x-}\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |\psi_{y-}\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad |0\rangle = |\psi_{z-}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  - Thus: spatial directions $\Longleftrightarrow$ pure states $\Longleftrightarrow$ $+1$-eigenstates of spin

- **Compound systems.** Pure states of compound systems are represented by unit vectors (rays) of the appropriate tensor product space:

$$\frac{1}{N}\sum_{ij} a_i b_j\, |\psi_i^1\rangle \otimes |\psi_j^2\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

where $\frac{1}{N}$ is the normalization coefficient. A pure state of the compound system is called entangled, if it is not product: $|\psi^1\rangle \otimes |\psi'^2\rangle$ (abbreviated as $|\psi^1\psi'^2\rangle$ or simply $|\psi\,\psi'\rangle$). An important entangled state is the singlet state: $|\psi^s\rangle = \frac{1}{\sqrt{2}}\big(\,|10\rangle - |01\rangle\,\big)$ with the corresponding projection:[5] $W^s = P^s = \frac{1}{2}\big(\,|10\rangle\langle 10| + |01\rangle\langle 01| - |10\rangle\langle 01| - |01\rangle\langle 10|\,\big)$

[5] $P^s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ where e.g. $|10\rangle\langle 10| = |1\rangle\langle 1| \otimes |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- **Three propositions on the singlet state:**
  - (i) The singlet state is rotationally invariant, it can be written as: $|\psi^s\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)$ for any directions
  - (ii) Correlation function: $E(\mathbf{a},\mathbf{b})^{|\psi^s\rangle} := \langle \mathbf{a}\boldsymbol{\sigma} \otimes \mathbf{b}\boldsymbol{\sigma}\rangle^{|\psi^s\rangle} = -\mathbf{ab}$
  - (iii) Quantum probabilities: $\mathrm{Tr}\big(W^s(P_{\mathbf{a}} \otimes P_{\mathbf{b}})\big) = \langle P_{\mathbf{a}} \otimes P_{\mathbf{b}}\rangle^{|\psi^s\rangle} = \frac{1}{2}\sin^2\left(\frac{\theta_{\mathbf{ab}}}{2}\right)$

Proofs:

(i) The effect of the rotation operator $e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}} = (\cos\theta\mathbf{1} + i\sin\theta\mathbf{n}\boldsymbol{\sigma})$ on $|1\rangle$ and $|0\rangle$ is:

$$\begin{aligned} e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}}\,|1\rangle &= (cos\theta + i\sin\theta\, n_z)\,|0\rangle + i\sin\theta\,(n_x + in_y)\,|1\rangle \\ e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}}\,|0\rangle &= i\sin\theta\,(n_x - in_y)\,|0\rangle + (cos\theta - i\sin\theta\, n_z)\,|1\rangle \end{aligned}$$

and therefore on the singlet state:

$$\begin{aligned} \left(e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}} \otimes e^{\frac{1}{2}i\theta\mathbf{n}\boldsymbol{\sigma}}\right)|\psi^s\rangle &= \frac{1}{\sqrt{2}}\Bigg[\big[\big((cos\theta + i\sin\theta\, n_z)\,|0\rangle + i\sin\theta\,(n_x + in_y)\,|1\rangle\,\big) \\ &\quad \otimes\big(i\sin\theta\,(n_x - in_y)\,|0\rangle + (cos\theta - i\sin\theta\, n_z)\,|1\rangle\,\big)\big] \\ &\quad -\big[\big(i\sin\theta\,(n_x - in_y)\,|0\rangle + (cos\theta - i\sin\theta\, n_z)\,|1\rangle\,\big) \\ &\quad \otimes\big((cos\theta + i\sin\theta\, n_z)\,|0\rangle + i\sin\theta\,(n_x + in_y)\,|1\rangle\,\big)\big]\Bigg] \\ &= \frac{1}{\sqrt{2}}\Bigg[\big[(cos\theta + i\sin\theta\, n_z)i\sin\theta\,(n_x - in_y) - (cos\theta + i\sin\theta\, n_z)i\sin\theta\,(n_x - in_y)\big]\,|0 \\ &\quad +\big[(cos\theta - i\sin\theta\, n_z)i\sin\theta\,(n_x + in_y) - (cos\theta - i\sin\theta\, n_z)i\sin\theta\,(n_x + in_y)\big]\,|11\rangle \\ &\quad +\big[(-\sin^2\theta\,(n_x^2 + n_y^2) - cos^2\theta - \sin^2\theta\, n_z^2)\big]\,|10\rangle \\ &\quad +\big[(cos^2\theta + \sin^2\theta\, n_z^2 + \sin^2\theta\,(n_x^2 + n_y^2))\big]\,|01\rangle\Bigg] \\ &= \frac{-1}{\sqrt{2}}\Big[(cos^2\theta + \sin^2\theta\,\mathbf{n}^2)\,|10\rangle - (cos^2\theta + \sin^2\theta\,\mathbf{n}^2)\,|01\rangle\Big] = -\,|\psi^s\rangle \end{aligned}$$

(ii) Using the rotational invariance of $|\psi^s\rangle$, taking the z-axis in direction $\mathbf{a}$ and the x-axis in the plane $\mathbf{ab}$, we obtain:

$$\begin{aligned} E(\mathbf{a},\mathbf{b})^{|\psi^s\rangle} &= \langle\psi^s|\big(\sigma_z \otimes (\cos\theta_{\mathbf{ab}}\sigma_z + \sin\theta_{\mathbf{ab}}\sigma_x)\big)\psi^s\rangle \\ &= \frac{1}{2}\big(\,\langle 10| - \langle 01|\,\big)|\big(\sigma_z \otimes (\cos\theta_{\mathbf{ab}}\sigma_z + \sin\theta_{\mathbf{ab}}\sigma_x)\big)\big(\,|10\rangle - |01\rangle\,\big) \\ &= \frac{1}{2}\big(\,\langle 10| - \langle 01|\,\big)|\big(-\cos\theta_{\mathbf{ab}}\,|10\rangle + \sin\theta_{\mathbf{ab}}\,|11\rangle - \cos\theta_{\mathbf{ab}}\,|01\rangle - \sin\theta_{\mathbf{ab}}\,|00\rangle\big) \\ &= -\cos\theta_{\mathbf{ab}} = -\mathbf{ab} \end{aligned}$$

where we used that $\sigma_x|1\rangle = |0\rangle$ and $\sigma_x|0\rangle = |1\rangle$.

(iii) First observe that $\langle 1|\mathbf{a}\boldsymbol{\sigma}|1\rangle = \langle 1|a_1\sigma_x + a_2\sigma_y + a_3\sigma_z|1\rangle = a_3$ and $\langle 0|\mathbf{a}\boldsymbol{\sigma}|0\rangle = -a_3$, and hence $\langle 10|\mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{1}|10\rangle + \langle 01|\mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{1}|01\rangle = a_3 - a_3 = 0$

$$\begin{aligned} \mathrm{Tr}\big(W^s(P_{\mathbf{a}} \otimes P_{\mathbf{b}})\big) &= \langle P_{\mathbf{a}} \otimes P_{\mathbf{b}}\rangle^{|\psi^s\rangle} = \langle\frac{1}{2}(\mathbf{1} + \mathbf{a}\boldsymbol{\sigma}) \otimes \frac{1}{2}(\mathbf{1} + \mathbf{b}\boldsymbol{\sigma})\rangle^{|\psi^s\rangle} \\ &= \frac{1}{4}\,\langle\psi^s|\big(\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{b}\boldsymbol{\sigma} + \mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{1} + \mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{b}\boldsymbol{\sigma}\big)\psi^s\rangle \\ &= \frac{1}{4}(1 + 0 + 0 - \mathbf{ab}) = \frac{1}{2}\left(\frac{1 - \mathbf{ab}}{2}\right) = \frac{1}{2}\sin^2\left(\frac{\theta_{\mathbf{ab}}}{2}\right) \end{aligned}$$

- **The EPR-Bohm scenario in QM.** According to the minimal interpretation (3)-(5) are represented in QM as

$$p(A_i|a_i) = \mathrm{Tr}\big(W^s(P_{\mathbf{a}_i} \otimes \mathbf{1})\big) \tag{10}$$

$$p(B_j|b_j) = \mathrm{Tr}\big(W^s(\mathbf{1} \otimes P_{\mathbf{b}_j})\big) \tag{11}$$

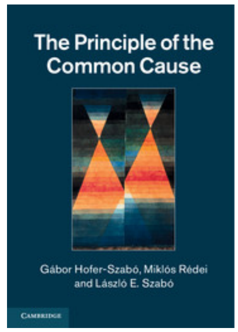
$$p(A_i \wedge B_j|a_i \wedge b_j) = \mathrm{Tr}\big(W^s(P_{\mathbf{a}_i} \otimes P_{\mathbf{b}_j})\big) \tag{12}$$

  yielding correct numerical values for the correlations. Note, however, that (10)-(12) do *not* causally account for these correlations; they simply describe them. Thus, the closing question of the last class remains: What is the causal explanation of these correlations?

3 The Common Cause Principle

Abstract: Reichenbach's Common Cause Principle will be introduced and illustrated on simple examples.

Literature: Hofer-Szabó, Rédei and Szabó, 2013, Ch. 1-5; Gyenis *et al.*, 2010



- **The Common Cause Principle (CCP)** states that if there is a correlation between two events A and B and there is no direct causal (or logical) connection between the correlating events, then there always exists a common cause C of the correlation. Reichenbach's merit is that he was the first to come up with a formal definition of a probabilistic common cause.

- **Reichenbachian common cause.** Let (Σ, p) be a classical probability measure space and let A and B be two positively correlating events in Σ that is let

  $$p(A \wedge B) > p(A)\, p(B). \tag{13}$$

  An event $C \in \Sigma$ is said to be the *(Reichenbachian) common cause* of the correlation between events A and B if the following conditions hold:

  $$p(A \wedge B|C) = p(A|C)p(B|C) \tag{14}$$
  $$p(A \wedge B|C^{\perp}) = p(A|C^{\perp})p(B|C^{\perp}) \tag{15}$$
  $$p(A|C) > p(A|C^{\perp}) \tag{16}$$
  $$p(B|C) > p(B|C^{\perp}) \tag{17}$$

  where $C^{\perp}$ denotes the orthocomplement of C and $p(\,\cdot\,|\,\cdot\,)$ is the conditional probability. Equations (14)-(15) are called *screening-off conditions*; inequalities (16)-(17) are called *positive statistical relevancy conditions.* (13) follows from (16)-(17).

  Conceptually, criteria (16)-(17) can be regarded only as necessary but not as sufficient conditions for an event to be a common cause. Still, in what follows they are taken to be *the* definition of the common cause.

- **Reichenbach's examples:**

> "Suppose both lamps in a room go out suddenly. We regard it as improbable that by chance both bulbs burned out at the same time and look for a burned out fuse or some other interruption of the common power supply. The improbable coincidence is thus explained as the product of a common cause." (p. 157)
>
> "Or suppose several actors in a stage play fall ill showing symptoms of food poisoning. We assume that the poisoned food stems from the same source – for instance, that it was contained in a common meal – and then look for an explanation of the coincidence in terms of a common cause." (p. 157)
>
> "Suppose two geysers which are not far apart spout irregularly, but throw up their columns of water always at the same time." (p. 158)

- **Common cause (system).** Reichenbach's definition is, however, too narrow for four reasons: First, the positive statistical relevancy conditions restrict one to common causes which increase the probability of their effects; or in other words, they exclude negative causes. Second, the definition also excludes situations in which the correlation is not due to a single cause but to a system of cooperating common causes. Third, it is silent about the spatiotemporal localization of the events. Fourth, it is classical.

  In this class we address only the first two problems. Let A and B be two correlating events in (Σ, p) that is

  $$p(A \wedge B) \neq p(A)\, p(B). \tag{18}$$

  A partition $\{C_k\}_{k \in K}$ in Σ is said to be the *common cause system* of the correlation (18) if the following screening-off condition holds for all $k \in K$:

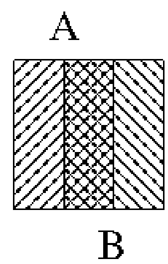
  $$p(A \wedge B | C_k) = p(A|C_k)\, p(B|C_k), \tag{19}$$

  where $|K|$, the cardinality of K is said to be the *size* of the common cause system. A common cause system of size 2 is called a common cause (without the adjective 'Reichenbachian'). A common cause system is called *trivial*, if $C_k \leq X$ with $X = A, A^{\perp}, B$ or $B^{\perp}$ for any $k \in K$.

- **Common cause extensions.** There exist probability spaces such that not all correlations have a common cause:

  Proposition 1. Every classical probability space (Σ, p) is common cause extendable with respect to any finite set of correlated events.

- **Common cause closedness.** Can we extend probability spaces such that *all* correlations have a common cause?



Proposition 2. (i) (Probabilistically) non-atomic probability measure spaces are common cause closed.

(ii) Atomic probability measure spaces are common cause closed iff they have exactly one atom.

(iii) All atomic probability measure spaces can be embedded into a common cause closed non-atomic probability measure space.

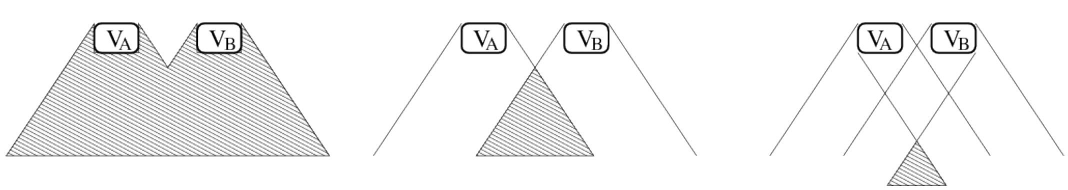
- **Joint common causes.** Let $\{(A_i, B_j); m \in M, n \in N\}$ be a set of correlating pairs in (Σ, p). A partition $\{C_k\}_{k\in K}$ in Σ is said to be the *joint common cause (system)* of the set if it screens off all correlations, that is for all $k \in K$:

$$p(A_i \wedge B_j | C_k) \;=\; p(A_i|C_k)\, p(B_j|C_k), \tag{20}$$

Probability spaces are typically not joint common cause extendable and not joint common cause closed.

- **Localization of the common cause.** Let V_A and V_B denote the two spacetime separated regions where the events A_i and B_j are localized. Relativistically these regions can have (at least) three different pasts: the *weak, common* and *strong* past of A and B as follows:

$$\begin{aligned}
\mathcal{P}^W(V_A, V_B) &:= I_-(V_A) \cup I_-(V_B)\\
\mathcal{P}^C(V_A, V_B) &:= I_-(V_A) \cap I_-(V_B)\\
\mathcal{P}^S(V_A, V_B) &:= \cap_{x\in V_A \cup V_B} I_-(x)
\end{aligned}$$



Call the appropriate common causes *weak common causes*, *common causes* and *strong common causes*, respectively. Note, however, that we do not possess yet a mathematical representation of the localization of the common causes!

4 The Bell inequalities

Abstract: The relation between the common causal explanation and the Bell inequalities will be explicated.

Literature: Hofer-Szabó, Rédei and Szabó, 2013, Ch. 9; Bell, 2004

- **Conditional correlations.** Consider the Bohm version of the EPR experiment as described in Section 2. As noted, there is a conditional correlation for any non-perpendicular directions $\mathbf{a}_i$ and $\mathbf{b}_j$:

$$p(A_i \wedge B_j | a_i \wedge b_j) \quad \neq \quad p(A_i|a_i)p(B_j|b_j) \tag{21}$$

  Let $i, j = \{1, 2\}$. What is the common causal explanation of these four correlations?

- **A local, non-conspiratorial joint common causal explanation** of the conditional correlations (21) consists in providing a single partition $\{C_k\}$ in Σ such that for any $i, i', j, j' = 1, 2$ the following requirements hold:

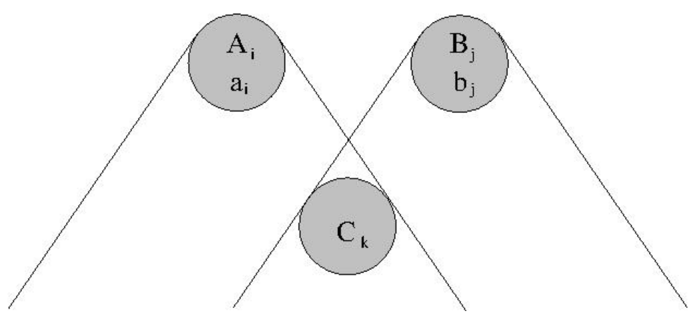
$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k) = p(A_i | a_i \wedge b_j \wedge C_k)\, p(B_j | a_i \wedge b_j \wedge C_k) \qquad \text{(screening-off)} \tag{22}$$

$$p(A_i | a_i \wedge b_j \wedge C_k) = p(A_i | a_i \wedge b_{j'} \wedge C_k) \qquad \text{(locality)} \tag{23}$$

$$p(B_j | a_i \wedge b_j \wedge C_k) = p(B_j | a_{i'} \wedge b_j \wedge C_k) \qquad \text{(locality)} \tag{24}$$

$$p(a_i \wedge b_j \wedge C_k) = p(a_i \wedge b_j)\, p(C_k) \qquad \text{(no-conspiracy)} \tag{25}$$

  The motivation behind requirements (22)-(25) is the following. *Screening-off* (22)



  is simply the application of the notion of common cause for conditional correlations: although A_i and B_j are correlating conditioned on a_i and b_j, they will cease to do so if we further condition on $\{C_k\}$. *Locality* (23)-(24) is the requirement that the measurement outcome on the one side should depend only on the measurement choice on the same side and the value of the common cause but not on the measurement choice on the opposite side. Finally, no-conspiracy (25) is the requirement that the common cause system and the measurement choices should be probabilistically independent.

  Note, however, that the spacetime localizations of the events is completely intuitive at this point. To proceed in a correct way, we need field theory.

Proposition 3. Let A_i, B_j, a_i and b_j $(i,j=1,2)$ be eight events in a classical probability measure space (Ω,Σ,p) such that the pairs $\{(A_i,B_j); i,j=1,2\}$ correlate in the conditional sense of (21). Suppose that $\{(A_i,B_j); i,j=1,2\}$ has a local, non-conspriratorial joint common causal explanation in the above sense. Then for any $i,i',j,j'=1,2; i\neq i'; j\neq j'$ the following *classical Clauser–Horne* inequality holds:

$$\begin{aligned}-1 \leqslant p(A_i\wedge B_j|a_i\wedge b_j) + p(A_i\wedge B_{j'}|a_i\wedge b_{j'}) + p(A_{i'}\wedge B_j|a_{i'}\wedge b_j) \\ -p(A_{i'}\wedge B_{j'}|a_{i'}\wedge b_{j'}) - p(A_i|a_i\wedge b_j) - p(B_j|a_i\wedge b_j) \leqslant 0 \end{aligned} \tag{26}$$

Proof. It is an elementary fact of arithmetic that for any $\alpha,\alpha',\beta,\beta'\in[0,1]$ the number

$$\alpha\beta+\alpha\beta'+\alpha'\beta-\alpha'\beta'-\alpha-\beta \tag{27}$$

lies in the interval $[-1,0]$. Now let $\alpha,\alpha',\beta,\beta'$ be the following conditional probabilities:

$$\alpha := p(A_i|a_i\wedge b_j\wedge C_k) \tag{28}$$
$$\alpha' := p(A_{i'}|a_{i'}\wedge b_{j'}\wedge C_k) \tag{29}$$
$$\beta := p(B_j|a_i\wedge b_j\wedge C_k) \tag{30}$$
$$\beta' := p(B_{j'}|a_{i'}\wedge b_{j'}\wedge C_k) \tag{31}$$

Plugging (28)-(31) into (27) and using locality (23)-(24) one obtains

$$\begin{aligned}-1 \leqslant p(A_i|a_i\wedge b_j\wedge C_k)p(B_j|a_i\wedge b_j\wedge C_k) + p(A_i|a_i\wedge b_{j'}\wedge C_k)p(B_{j'}|a_i\wedge b_{j'}\wedge C_k) \\ +p(A_{i'}|a_{i'}\wedge b_j\wedge C_k)p(B_j|a_{i'}\wedge b_j\wedge C_k) - p(A_{i'}|a_{i'}\wedge b_{j'}\wedge C_k)p(B_{j'}|a_{i'}\wedge b_{j'}\wedge C_k) \\ -p(A_i|a_i\wedge b_j\wedge C_k) - p(B_j|a_i\wedge b_j\wedge C_k) \leqslant 0\end{aligned} \tag{32}$$

Using screening-off (22) one gets

$$\begin{aligned}-1 \leqslant p(A_i\wedge B_j|a_i\wedge b_j\wedge C_k) + p(A_i\wedge B_{j'}|a_i\wedge b_{j'}\wedge C_k) + p(A_{i'}\wedge B_j|a_{i'}\wedge b_j\wedge C_k) \\ -p(A_{i'}\wedge B_{j'}|a_{i'}\wedge b_{j'}\wedge C_k) - p(A_i|a_i\wedge b_j\wedge C_k) - p(B_j|a_i\wedge b_j\wedge C_k) \leqslant 0\end{aligned} \tag{33}$$

Multiplying the above inequality by $p(C_k)$, using no-conspiracy (25) and summing up for the index k one obtains

$$\begin{aligned}-1 \leqslant \sum_k \big(p(A_i\wedge B_j\wedge C_k|a_i\wedge b_j) + p(A_i\wedge B_{j'}\wedge C_k|a_i\wedge b_{j'}) + p(A_{i'}\wedge B_j\wedge C_k|a_{i'}\wedge b_j) \\ -p(A_{i'}\wedge B_{j'}\wedge C_k|a_{i'}\wedge b_{j'}) - p(A_i\wedge C_k|a_i\wedge b_j) - p(B_j\wedge C_k|a_i\wedge b_j)\big) \leqslant 0\end{aligned} \tag{34}$$

Finally, applying the theorem of total probability

$$\sum_k p(Y\wedge C_k) = p(Y)$$

one arrives at (26) which completes the proof. □

- **QM violates the Clauser-Horne inequality.** Measure the spin of particle a in direction $\mathbf{a}_1$ or $\mathbf{a}_2$ and that of particle b in direction $\mathbf{b}_1$ or $\mathbf{b}_2$ where

$$\sphericalangle(\mathbf{a}_1, \mathbf{b}_1) = \sphericalangle(\mathbf{a}_1, \mathbf{b}_2) = \sphericalangle(\mathbf{a}_2, \mathbf{b}_1) = 120^\circ \text{ and } \sphericalangle(\mathbf{a}_2, \mathbf{b}_2) = 0^\circ$$

Then the Clauser-Horne inequality

$$\begin{aligned} -1 \leqslant p(A_1 \wedge B_1 | a_1 \wedge b_1) + p(A_1 \wedge B_2 | a_1 \wedge b_2) + p(A_2 \wedge B_1 | a_2 \wedge b_1) \\ -p(A_2 \wedge B_2 | a_2 \wedge b_2) - p(A_1 | a_1 \wedge b_1) - p(B_1 | a_1 \wedge b_1) = \frac{1}{8} \nleqslant 0 \end{aligned} \quad (35)$$

is violated excluding a local, non-conspiratorial joint common causal explanation of the conditional correlations (21).

- **A local, non-conspiratorial separate common causal explanation** of the conditional correlations (21) consists in providing four partitions $\{C_k^{ij}\}$ in Σ such that for any $i, i', j, j' = 1, 2$ the following requirements hold:

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k^{ij}) = p(A_i | a_i \wedge b_j \wedge C_k^{ij})\, p(B_j | a_i \wedge b_j \wedge C_k^{ij}) \quad \text{(screening-off)} \quad (36)$$

$$p(A_i | a_i \wedge b_j \wedge C_k^{ij}) = p(A_i | a_i \wedge b_{j'} \wedge C_k^{ij}) \quad \text{(locality)} \quad (37)$$

$$p(B_j | a_i \wedge b_j \wedge C_k^{ij}) = p(B_j | a_{i'} \wedge b_j \wedge C_k^{ij}) \quad \text{(locality)} \quad (38)$$

$$p(a_i \wedge b_j \wedge D) = p(a_i \wedge b_j)\, p(D) \quad \text{(no-conspiracy)} \quad (39)$$

where D is an element of the algebra generated by all separate common cause systems.

- **Open problem.** It is not known whether the Clauser-Horne inequalities (26) can be derived from the local, non-conspiratorial separate common causal explanation (36)-(39).

5 The algebraic approach to quantum theory

Abstract: The basics of the theory of C^*-algebras and von Neumann algebras will be presented.

Literature: Halvorson, 2007; Earman and Ruetsche, 2011; Rédei, 1995; Rédei and Summers, 2007.

- **Topologies**. There are four standard topologies in $\mathcal{B}(\mathcal{H})$:

  1. The *uniform (norm) topology* is defined by a single norm

  $$\|A\| = \sup\{\|A\,|\phi\rangle\,\| : |\phi\rangle \in \mathcal{H},\ \|\,|\phi\rangle\,\| = 1\} \tag{40}$$

  A sequence $\{A_i\}$ *uniformly converges* to A, $\{A_i\} \xrightarrow{u} A$ iff $\{\|A_i - A\|\} \longrightarrow 0$.

  2. The *strong topology* is defined by the family $\{p_\phi : |\phi\rangle \in \mathcal{H}\}$ of seminorms where

  $$p_\phi(A) = \|A\,|\phi\rangle\,\| \tag{41}$$

  A sequence $\{A_i\}$ *strongly converges* to A, $\{A_i\} \xrightarrow{s} A$ iff $\{p_\phi(A_i)\} \longrightarrow p_\phi(A)$ for all $|\phi\rangle \in \mathcal{H}$.

  3. The *ultraweak topology* is defined by the family $\{p_W : 0 \leqslant W^* = W \in \mathcal{B}(\mathcal{H}),\ \mathrm{Tr}(W) = 1\}$ where

  $$p_W(A) = \mathrm{Tr}(WA) \tag{42}$$

  A sequence $\{A_i\}$ *ultraweakly converges* to A, $\{A_i\} \xrightarrow{uw} A$ iff $\{p_W(A_i)\} \longrightarrow p_W(A)$ for all density operators.

  4. The *weak topology* is defined by the family $\{p_{\phi,\psi} : |\phi\rangle, |\psi\rangle \in \mathcal{H}\}$ of seminorms where

  $$p_{\phi,\psi}(A) = \langle\psi|A|\phi\rangle \tag{43}$$

  A sequence $\{A_i\}$ *weakly converges* to A, $\{A_i\} \xrightarrow{w} A$ iff $\{p_{\phi,\psi}(A_i)\} \longrightarrow p_{\phi,\psi}(A)$ for all $|\phi\rangle$ and $|\psi\rangle \in \mathcal{H}$.

  A topology is stronger if fewer sequences converge. For example, countable sequences of pairwise orthogonal projections converge strongly but not uniformly. The implication between the topologies is this: norm $\Rightarrow$ ultraweak $\Rightarrow$ weak; norm $\Rightarrow$ strong $\Rightarrow$ weak. The four topologies coincide iff $\mathcal{H}$ is finite dimensional.

- **Jordan algebra**. Self-adjoint elements of $\mathcal{B}(\mathcal{H})$ representing observables are not close under multiplication: unless $A, B \in \mathcal{B}(\mathcal{H})$ commute, their product AB is not self-adjoint. One way to preserve the closed product structure of self-adjoint operators is to introduce the *Jordan product*: $\{A, B\} := \frac{1}{2}(AB + BA)$. The Jordan product is *not associative*. The self-adjoint part of $\mathcal{B}(\mathcal{H})$ is a *Jordan algebra* (or, if equipped with a norm, a Jordan-Banach algebra).

- **A C^*-algebra** $\mathcal{A}$ is an algebra equipped with an *involution* $\mathcal{A} \to \mathcal{A} : A \mapsto A^*$ satisfying $(A^*)^* = A$, $(A+B)^* = A^* + B^*$, $(cA)^* = \bar{c}A^*$ and $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$ and all $c \in \mathbb{C}$ ($\bar{c}$ is for the complex conjugate of c) and also equipped with a *norm* satisfying $\|A^*A\| = \|A\|^2$ and $\|AB\| \leqslant \|A\|\|B\|$ for all $A, B \in \mathcal{A}$, and which is complete in the topology induced by that norm. In the concrete form a A C^*-algebra is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the norm topology. According to Gelfand's representation theorem every abelian C^*-algebra is isomorphic to $C(X)$, the set of continuous complex-valued functions on X for some compact Hausdorff space X. Classical/quantum observables are the self-adjoint elements of an abelian/non-abelian, unital C^*-algebra.

- **A state** ϕ on a C^*-algebra $\mathcal{A}$ is a normalized, positive linear map $\phi : \mathcal{A} \mapsto C$, where *normalization* means that $\phi(\mathbf{1}) = 1$ and *positivity* means that $\phi(A^*A) \geqslant 0$ for any $A \in \mathcal{A}$ (an element B of $\mathcal{A}$ is positive if there is an $A \in \mathcal{A}$ such that $B = A^*A$). A state is *mixed* if it is a nontrivial linear combination of other states, otherwise it is *pure*. A state ϕ on $\mathcal{A}$ is *faithful* just in case $\phi(A^*A) > 0$ for any nonzero $A \in \mathcal{A}$, and *tracial* if $\phi(AB) = \phi(BA)$ for any nonzero $A, B \in \mathcal{A}$.

- **A von Neumann algebra** $\mathcal{N}$ is a C^*-algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$ which is closed in the weak topology of this space. A sequence of bounded operators $\{A_i\}$ converges in the *weak topology* to A just in case $\langle\psi|A_i|\psi'\rangle$ converges to $\langle\psi|A|\psi'\rangle$ for all $|\psi\rangle, |\psi'\rangle \in \mathcal{H}$. Von Neumann's *double commutant theorem* states that $\mathcal{N}$ is weakly closed iff $\mathcal{N}'' = \mathcal{N}$, where $\mathcal{N}'$ denotes the commutant of $\mathcal{N}$. The *center* $Z(\mathcal{N})$ of a von Neumann algebra is $\mathcal{N} \cap \mathcal{N}'$. $\mathcal{N}$ is a *factor* algebra if its center is trivial, $\mathcal{N} \cap \mathcal{N}' = \mathbb{C}\mathbf{1}$.

  A paradigmatic abelian von Neumann algebra is $L^\infty(X, \Sigma, \mu)$, the space of complex-valued essentially bounded measurable functions on the σ-finite measure space (X, Σ, μ) acting on the separable Hilbert space $L^2(X, \Sigma, \mu)$ by multiplication. While there are no projections in the C^*-algebra $C(X)$ at all, the von Neumann algebra $L^\infty(X, \Sigma, \mu)$ is generated by its projections, the characteristic functions $\{\chi_S, S \in \Sigma\}$ on X. Any (normal) state ω on the von Neumann algebra $L^\infty(X, \Sigma, \mu)$ determines a probability measure p_ϕ on the σ-algebra (X, Σ) by $p_\phi(S) := \omega(\chi_S), S \in \Sigma$.

- **GNS-representation**. A *representation* of a C^*-algebra $\mathcal{A}$ is a **-homomorphism* $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. The representation π is *faithful* if it is a *-isomorphism, and *irreducible* if there is no nontrivial subspace of $\mathcal{H}$ which is invariant under $\pi(\mathcal{A})$. GNS-representation: C^*-algebraic states can be represented as vectors of a Hilbert space: any state ϕ on $\mathcal{A}$ determines (up to unitary equivalence) a triple $(\pi_\phi, \mathcal{H}_\phi, |\Omega_\phi\rangle)$ such that $\phi(A) = \langle\Omega_\phi|\pi_\phi(A)|\Omega_\phi\rangle$ for all $A \in \mathcal{A}$, and the representation $\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ is *cyclic* with respect to the vector $|\Omega_\phi\rangle \in \mathcal{H}_\phi$, that is $\{\pi_\phi(\mathcal{A})\,|\Omega_\phi\rangle\}$ is dense in $\mathcal{H}_\phi$.

  A central theorem about the GNS-representation: a state ϕ is pure iff π_ϕ is irreducible. One can associate a von Neumann algebra $\mathcal{N}$ with a representation π of

$\mathcal{A}$ as $\mathcal{N} := (\pi(\mathcal{A}))''$. If π is irreducible (as is the case with the GNS-representation induced by a pure state), $(\pi(A))' = \mathbb{C}\mathbf{1}$ (otherwise the projections onto a nontrivial invariant subspace would belong to $(\pi(A))'$), and $(\pi(A))'' = \mathcal{B}(\mathcal{H})$.

- **Normal and vector states**. A state ϕ is *normal* on a von Neumann algebra $\mathcal{N}$ acting on $\mathcal{H}$ iff there is a *density operator* W $(0 \leqslant W^* = W \in \mathcal{B}(\mathcal{H})$ and $\mathrm{Tr}(W) = 1)$ on H such that $\phi(A) = \mathrm{Tr}(WA)$ for all $A \in \mathcal{N}$. Equivalently, a state ϕ is normal, if it is *σ-additive*, i.e. $\phi(\sum_n P_n) = \sum_n \phi(P_n)$ for any countable set of pairwise orthogonal projections in $\mathcal{N}$. Every normal state on $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is the restriction of a normal state on $\mathcal{B}(\mathcal{H})$. Hence, for any normal state on $\mathcal{N}$ there exists a (no longer necessarily unique) density matrix in $\mathcal{B}(\mathcal{H})$ (!) representing the state in the above sense.

  Every normal state ϕ on $\mathcal{N}$ determines a σ-additive probability measure: $p : \mathcal{P}(\mathcal{N}) \to [0,1], P \mapsto \phi(P)$, where $\mathcal{P}(\mathcal{N})$ is the set of projections in $\mathcal{N}$. Conversely, for any σ-additive probability measure p on any $\mathcal{P}(\mathcal{N})$ (not containing a summand of type I_2), there is a unique normal state ϕ on $\mathcal{N}$ such that $p(P) = \phi(P)$ for all $P \in \mathcal{P}(\mathcal{N})$ (generalized Gleason's theorem).

  A *vector state* ϕ for a von Neumann algebra $\mathcal{N}$ acting on $\mathcal{H}$ is a state such that there is a $|\phi\rangle \in \mathcal{H}$ and a corresponding projection P_ϕ (projecting onto the one-dimensional subspace of $\mathcal{H}$ spanned by $|\phi\rangle$) with $\phi(A) = \langle\phi|A|\phi\rangle = \mathrm{Tr}(P_\phi A)$ for all $A \in \mathcal{N}$.

- **Folium**. Two states ϕ and ϕ' are called *unitarily equivalent* if their GNS-representation π_ϕ and $\pi_{\phi'}$ are unitarily equivalent; they are called *disjoint* if no subrepresentation of π is unitarily equivalent to a subrepresentation of π'; and they are called *quasi-equivalent* if π_ϕ and $\pi_{\phi'}$ are quasi-equivalent that is *-isomorphic (as for example π_ϕ and $\pi_\phi \oplus \pi_\phi$). Quasi-equivalence is an equivalence relation on the set of representations.

  Disjointness of algebraic states radicalizes orthogonality: for disjoint states no state possible for the one is possible for the other (where "ϕ is possible for ϕ'" is understood that the transition probability $1 - \frac{1}{4}\|\phi - \phi'\|$ is nonzero). Unitary equivalence implies quasi-equivalence. Pure states (irreducible GNS-representations) are either unitarily equivalent or disjoint, but mixed states can also be quasi-equivalent.

  A state ϕ is *π-normal* iff it can be represented by a normal state in the representation π. The *folium* $\mathcal{F}_\omega$ of a representation π_ϕ of $\mathcal{A}$ is the set of π_ϕ-normal states. If π_ϕ and $\pi_{\phi'}$ are quasi-equivalent, then $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$; if π_ϕ and $\pi_{\phi'}$ are disjoint, then $\mathcal{F}_\omega \cap \mathcal{F}_{\omega'} = \emptyset$. Fell's theorem claims that if π_ϕ is faithful, then $\mathcal{F}_\omega$ is dense (in the weak topology) in the state space of the C^*-algebra.

- **Lattices** can be defined in two different ways. First, a *lattice* is a partially ordered set $\mathcal{L} = (\mathcal{S}, \leqslant)$ such that for any two element A and B there exist the least upper bound $sup\{A, B\}$ and the greatest lower bound $inf\{A, B\}$. Second, a lattice is an algebraic structure $\mathcal{L} = (\mathcal{S}, \wedge, \vee)$ where the operations are commutative, associative, idempotent and fulfill the absorption law ($A \wedge (A \vee B) = A$). The two definitions can be made equivalent by $A \wedge B = inf\{A, B\}$ and $A \vee B = sup\{A, B\}$.

A lattice is *complete* if any subset has a greatest lower and a least upper bound and is a *σ-lattice* if they exist for any countable subset. Usually it is assumed that a lattice has a smallest element, $\mathbf{0}$, and a greatest element, $\mathbf{1}$. The element $A \in \mathcal{L}$ is an *atom* in $\mathcal{L}$ if $B \leqslant A$ implies $B = A$ or $B = 0$. The lattice $\mathcal{L}$ is called an *atomic* lattice if for any $B \in \mathcal{L}$ there exists an atom A such that $A \leqslant B$. The lattice is called *completely atomistic* if any element is equal to the least upper bound of all the atoms it majorizes.

On a lattice one can introduce an operation called *orthocomplementation* by (i) $(A^{\perp})^{\perp} = A$, (ii) $A \leqslant B$ iff $B^{\perp} \leqslant A^{\perp}$, (iii) $A \wedge A^{\perp} = 0$ and (iv) $A \vee A^{\perp} = 1$. In an orthocomplemented lattice the *De Morgan identities* fulfil.

Observe that *distributivity* does not feature among the characterizing properties of a lattice. A lattice is called *modular* if $P \leqslant Q$ implies $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) = Q \wedge (P \vee R)$ and called *orthomodular* if $P \leqslant Q$ implies $P \vee (P^{\perp} \wedge Q) = Q$. The (strict) implication between these properties is this: distributive $\Rightarrow$ modular $\Rightarrow$ orthomodular.

One can investigate lattices *via* their *Hasse diagrams*. A lattice is modular only if N_5 cannot be embedded in it, and distributive only if neither N_5 nor M_5 can be embedded in it.

- **Projection lattice**. From the double commutant theorem it follows that $\mathcal{P}(\mathcal{N})$ is a *complete*, *orthomodular lattice* and that it determines $\mathcal{N}$ completely in the sense that $\mathcal{P}(\mathcal{N})'' = \mathcal{N}$. Hence, it is enough to investigating the lattice $\mathcal{P}(\mathcal{N})$ to get information from the algebra itself. Two projections P and Q in $\mathcal{N}$ are called *equivalent*, $P \sim Q$, with respect to the algebra $\mathcal{N}$ if there is an operator, a partial isometry in $\mathcal{N}$ (!) which maps the range of $\mathbf{1} - P$ onto $\mathbf{0}$ and is an isometry between the ranges of P and Q. By means of the equivalence relation $\sim$ one can define a *partial ordering* on $\mathcal{P}(\mathcal{N})$: $P \preccurlyeq Q$ iff there exists a $P' \in \mathcal{P}(\mathcal{N})$ such that $P \sim P' \leqslant Q$.

- **Finite, abelian and minimal projections**. A projection $P \in \mathcal{P}(\mathcal{N})$ is *infinite* if there is a Q such that $Q < P$ and $Q \sim P$, otherwise is *finite*. A nonzero projection P is *abelian* iff the von Neumann algebra $P\mathcal{N}P$ (in which P serves as the identity), acting on the Hilbert space $P\mathcal{N}$, is abelian. A nonzero projection P is *minimal* iff P's only subprojections are $\mathbf{0}$ and P itself. The implication between these properties is this: minimal $\Rightarrow$ abelian $\Rightarrow$ finite, but in general the arrows cannot be reversed.

- **Classification of von Neumann factors**. If $\mathcal{N}$ is a factor, then $\mathcal{P}(\mathcal{N})_{\sim}$, the set of equivalence classes is totally ordered with respect to $\sim$. Two factors are isomorphic iff their projection lattices are isomorphic with respect to the ordering $\preccurlyeq$. Murray and von Neumann based the *classification* of von Neumann factors on this fact. For any factor $\mathcal{N}$ there exists a map $d : \mathcal{P}(\mathcal{N}) \to [0, \infty]$, the *dimension function* with the following properties:

(i) $d(P) = 0$ iff $P = 0$,

(ii) if $P \perp Q$, then $d(P + Q) = d(P) + d(Q)$,

(iii) $d(P) \leqslant d(Q)$ iff $P \preccurlyeq Q$,

(iv) $d(P) < \infty$ iff P is a finite projection,

(v) $d(P) = d(Q)$ iff $P \sim Q$,

(vi) $d(P) + d(Q) = d(P \wedge Q) + d(P \vee Q)$.

The order type of $\mathcal{P}(\mathcal{N})_\sim$ can be characterized by the range of the dimension function:

Range of d	Type of factor $\mathcal{N}$	Property of the range
$\{0, 1 \ldots n\}$	I_n	discrete, finite
$\{0, 1 \ldots \infty\}$	I_∞	discrete, infinite
$[0, 1]$	II_1	continuous, finite
$[0, \infty]$	II_∞	continuous, infinite
$\{0, \infty\}$	III	purely infinite

If $\mathcal{N}$ is finite, then its projection lattice $\mathcal{P}(\mathcal{N})$ is *modular*; if $\mathcal{N}$ is infinite, then $\mathcal{P}(\mathcal{N})$ is *orthomodular* but not modular. There exists a *faithful normal tracial state* on a factor $\mathcal{N}$ iff it is finite.

- **Type I factors** contain *minimal projections* (hence abelian and finite projections). The algebra $\mathcal{B}(\mathcal{H}_n)$ is of type I_n, and the algebra $\mathcal{B}(\mathcal{H})$ is of I_∞. Type I factors play a role in non-relativistic quantum mechanics.
- **Type II factors** contain *no abelian projections* (hence no finite projections), but contain (nonzero) *finite* projections. Type II_1 factors have projections whose ranges are subspaces of *fractional* dimension. The identity operator in a factor of type II_1 is finite, and infinite in a factor of type II_∞. The infinite spin chain provides an example of type II_1 factor. It is historically interesting that von Neumann, the founder of the Hilbert space formalism of QM, gradually became convinced that to the correct quantum logic of QM is the type II_1 factor, since its projection lattice is modular (as opposed to that of $\mathcal{B}(\mathcal{H})$ which is only orthomodular), and considered modularity to be necessary to define an a priori probability *via* the trace.
- **Type III factors** contain *no (nonzero) finite projections* (so neither minimal nor abelian projections). All their projections are infinite and therefore equivalent. For any projection $P \in \mathcal{N}$ there exist countably infinitely many mutually orthogonal projections $P_i \in \mathcal{N}$ such that $P = \vee_i P_i$. Type III factors are used in algebraic quantum field theory and quantum statistical mechanics. Type

III factors can be further subdivided into type III_λ algebras, $\lambda \in [0, 1]$, by the Tomita-Takesaki modular theory.

Any von Neumann algebra can be decomposed into a direct sum of the above factors.

If $\mathcal{N}$ is atomless (as is the case with type II and III factors), it admits *no pure normal states*. Vector states are normal; consequently, for type II and III algebras no vector state is pure. Normal states are vector states if $\mathcal{N}$ has a separating vector. A vector $|\phi\rangle \in \mathcal{H}$ is a *separating* if the only $A \in \mathcal{N}$ satisfying $A|\phi\rangle = 0$ is the $A = 0$. ($|\phi\rangle$ is separating for $\mathcal{N}$ iff it is cyclic for $\mathcal{N}'$.[6]) Type I von Neumann algebras do not have separating vectors, therefore normal states are not necessarily vector states. Type III von Neumann algebras do have separating vectors, therefore normal states are vector states.

[6]If $|\phi\rangle$ is non-separating for $\mathcal{N}$, then there is a non-zero $A \in \mathcal{N}$ such that $A|\phi\rangle = 0$. Since for any $B \in \mathcal{N}' : AB|\phi\rangle = BA|\phi\rangle = 0$, no vector $|\phi'\rangle \in \mathcal{H}$ with $A|\phi'\rangle \neq 0$ can be obtained as $|\phi'\rangle = B|\phi\rangle$. That is $|\phi\rangle$ is not cyclic for $\mathcal{N}'$.

If $|\phi\rangle$ is not cyclic for $\mathcal{N}'$, then there exist a $|\phi'\rangle \in \mathcal{H}$ such that $|\phi'\rangle \neq B|\phi\rangle$ for any $B \in \mathcal{N}'$. Moreover, $|\phi'\rangle$ can also be chosen to be perpendicular to $|\phi\rangle$. Then for the projection $P_{|\phi'\rangle} \in \mathcal{N}''(= \mathcal{N}) : P_{|\phi'\rangle}|\phi\rangle = 0$, hence $|\phi\rangle$ is non-separating for $\mathcal{N}$.

6 What is a local physical theory?

Abstract: The net of local observable algebras will be introduced.

Literature: Haag, 1992; Halvorson, 2007; Ruetsche, 2012; Earman and Valente, 2014; Hofer-Szabó and Vecsernyés, 2015a,b.

- **Spacetime**. The central idea of a local physical theory, be it classical or quantum, is the association of local operator algebras to spacetime regions. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. A *spacetime* is a connected time-oriented Lorentzian manifold. A spacetime $(\mathcal{M}, g)$ is *globally hyperbolic* if $\mathcal{M}$ contains a Cauchy hypersurface, a subset $\mathcal{C} \subset \mathcal{M}$ such that each inextendible timelike curve in $\mathcal{M}$ meets $\mathcal{C}$ at exactly one point. Let $\mathcal{K}$ be a *covering collection*[7] of bounded, globally hyperbolic subspacetime regions of $\mathcal{M}$ such that $(\mathcal{K}, \subseteq)$ is a directed poset under inclusion $\subseteq$.

- **Isotony**. The *net* $\{\mathcal{A}(V), V \in \mathcal{K}\}$ of local observables is given by the isotone map $\mathcal{K} \ni V \mapsto \mathcal{A}(V)$ to unital C^*-algebras, that is $V_1 \subseteq V_2$ implies that $\mathcal{A}(V_1)$ is a unital C^*-subalgebra of $\mathcal{A}(V_2)$. Isotony expresses the idea that if an observable is measurable in a region V_1, then it is also measurable in a bigger region V_2 containing V_1. The *quasilocal algebra* $\mathcal{A}$ is defined to be the inductive limit C^*-algebra generated by the local algebras of the net: $\mathcal{A} := \overline{\cup_{V \in \mathcal{K}} A(V)}$, where the closure is taken in the C^*-norm. Sometimes *additivity*, which is a stronger property than isotony, is also required: $\mathcal{A}(V_1) \vee \mathcal{A}(V_2) = \mathcal{A}(V_1 \cup V_2); V_1, V_2, V_1 \cup V_2 \in \mathcal{K}$, where $\vee$ refers to the generated algebra in $\mathcal{A}$.

- **Microcausality** (also called as *Einstein locality*) is the requirement that $\mathcal{A}(V')' \cap \mathcal{A} \supseteq \mathcal{A}(V), V \in \mathcal{K}$, where primes denote spacelike complement and algebra commutant, respectively. Microcausality fulfills trivially in local classical theories. Note that field operators need not satisfy microcausality. Microcausality can be motivated by the *no-signalling theorem* (see below) stating that if microcausality holds, then non-selective measurements in spatially separated regions do not disturbe each other. But this no-disturbance-at-a-distance-argument for microcausality works only for non-selective operations.

- **$\mathcal{P}_{\mathcal{K}}$-covariance.** A diffeomorphism Φ of $(\mathcal{M}, g)$ is called an *isometry* if $\Phi^* g = g$, where $\Phi^* g$ is the "push-forward" of g by Φ. Let $\mathcal{P}$ be the isometry group of $(\mathcal{M}, g)$ and let $\mathcal{P}_{\mathcal{K}}$ be the subgroup of $\mathcal{P}$ leaving the collection $\mathcal{K}$ invariant. $\mathcal{P}_{\mathcal{K}}$-covariance means that there is a group homomorphism $\alpha\colon \mathcal{P}_{\mathcal{K}} \to \mathrm{Aut}\,\mathcal{A}$ such that the automorphisms $\alpha_g, g \in \mathcal{P}_{\mathcal{K}}$ of $\mathcal{A}$ act covariantly on the observable net: $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(g \cdot V), V \in \mathcal{K}$. $\mathcal{P}_{\mathcal{K}}$-covariance demands that if two spacetime regions are connected by a spacetime symmetry, then the associated local algebras should be isomorphic.

- **Algebraic Haag duality** is a strengthening of microcausality: $\mathcal{A}(V')' \cap \mathcal{A} = \mathcal{A}(V), V \in \mathcal{K}$. It is inherently connected to the noncommutativity of $\mathcal{A}$. In case

[7] For all $x \in \mathcal{M}$ there exists $V \in \mathcal{K}$ such that $x \in V$.

of commutative $\mathcal{A}$ instead of Haag duality one requires less, namely *intersection property for spacelike separated regions.* The intersection property $\mathcal{A}(V_1) \cap \mathcal{A}(V_2) = \mathcal{A}(V_1 \cap V_2); V_1, V_2, V_1 \cap V_2 \in \mathcal{K}$ holds for spacelike separated regions $V_1, V_2 \in \mathcal{K}$, that is $\mathcal{A}(V_1) \cap \mathcal{A}(V_2) = \mathcal{A}(\emptyset) := \mathbb{C}\,\mathbf{1}_{\mathcal{A}}$ for them.

- **Representation**. $\mathcal{P}_{\mathcal{K}}$-covariance does not mean that any state ϕ is *α-invariant*, $\phi \circ \alpha_g = \phi$ for any $g \in \mathcal{P}_{\mathcal{K}}$. A state ϕ *unitarily implements* α if in the (locally faithful) GNS-representation π_ϕ there is a (strongly continuous) unitary representation $U \colon \mathcal{P}_{\mathcal{K}} \to \mathcal{B}(\mathcal{H})$ of α, that is

$$\pi_\phi(\alpha_g(A)) = U(g)\pi_\phi(A)U(g)^*, \qquad A \in \mathcal{A},\ g \in \mathcal{P}_{\mathcal{K}}. \tag{44}$$

  The representations is faithful not to loose local observables. By taking weak closures $\mathcal{N}(V) := \pi(\mathcal{A}(V))''$, $V \in \mathcal{K}$ and $\mathcal{A}_{\mathcal{H}} := \overline{\cup_{V \in \mathcal{K}} \mathcal{N}(V)} \subset \mathcal{B}(\mathcal{H})$ one can consider the natural von Neumann algebra extension of the local algebras. If ϕ is α-invariant, then it also unitarily implements α.

- **A local physical theory** (LPT) is a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local von Neumann algebras associated to a directed poset $\mathcal{K}$ of globally hyperbolic bounded regions of a globally hyperbolic spacetime $\mathcal{M}$. The net satisfies *isotony*, *microcausality*, *$\mathcal{P}_{\mathcal{K}}$-covariance*, and *intersection property for spacelike separated regions.* If the quasi-local algebra $\mathcal{A}$ is commutative, we speak about a *local classical theory* (LCT), if it is noncommutative, we speak about a *local quantum theory* (LQT).

- **Algebraic quantum field theory** (AQFT) is a LQT with some extra axioms. Here the spacetime $\mathcal{M}$ is the Minkowski spacetime, $\mathcal{K}$ is the net of all double cones, and $\mathcal{P}_{\mathcal{K}} = \mathcal{P}$ is the Poincaré group. A *double cone* in $\mathcal{M}$ is the intersection of the causal past of a point x with the causal future of a point y timelike to x.

  One then introduces further requirements on the representations of $\mathcal{A}$:

  (i) *Vacuum condition.* There is a (up to a scalar) unique vector Ω in the Hilbert space $\mathcal{H}_0$ corresponding to the vacuum state ϕ_0 such that $U(g)\Omega = \Omega$ for all $g \in \mathcal{P}$.

  (ii) *Spectrum condition.* The spectrum of the self-adjoint generators of the strongly continuous unitary representation of the translation subgroup $\mathbb{R}^4$ of $\mathcal{P}$ lies in the closed forward light cone.

  (iii) *Weak additivity.* For any nonempty open region V, the set of operators $\cup_{g \in \mathbb{R}^4} \mathcal{N}(g \cdot V)$ is dense in $\mathcal{B}(\mathcal{H}_0)$ (in the weak operator topology).

  (iv) *The type of the algebras.* For every double cone V the von Neumann algebra $\mathcal{N}(V)$ is of type III_1 hyperfinite factors, where an algebra is hyperfinite if it is weak closure of an ascending sequence of finite dimensional algebras.

- **The Reeh–Schlieder Theorem** is an immediate consequence of above assumptions: For any nonempty open region V, Ω is cyclic in $\mathcal{H}_0$ that is the set of vectors

$\mathcal{N}(V)\Omega$ is dense in $\mathcal{H}_0$. Crudely, one can produce arbitrary approximations of any global state by local operations.

Since Ω is also separating for each $\mathcal{N}(V)$, every local event has a nonzero probability of occurring in the vacuum state (since $P\Omega \neq 0$ for any $\mathbf{0} \neq P \in \mathcal{N}(V)$, therefore $\|P\Omega\| \neq 0$), and there are no local number operators.

- **Abstract of concrete?**

  1. *"Algebraic imperialism"*. The physical content of an AQFT is encoded in the *abstract net* $\{\mathcal{A}(V), V \in \mathcal{K}\}$ together with the subgroup of $\operatorname{Aut}\mathcal{A}$ corresponding to physical symmetries (including dynamics), and the states on $\mathcal{A}$.
  2. *"Hilbert space conservatism"*. The physical content of an AQFT is encoded in a representation π of the net. Two theories are equivalent if they are unitarily equivalent. The goal here is to introduce some physical requirements (unitarily implementability of the spacetime symmetries, vacuum condition, Hadamard condition, etc.) in order to pick one of the many inequivalent representations.
  3. *DHR superselection theory.* "Physical" representations are those that differ from the vacuum representation only locally. These (DHR) representations correspond to the category Δ of localized transportable endomorphisms of $\mathcal{A}$.

     An *endomorphism* $\rho : \mathcal{A} \to \mathcal{A}$ is a (not necessarily surjective) *-homomorphism. ρ is *localized* in a double cone V if $\rho(A) = A$, for all $A \in \mathcal{A}(V')$. A ρ localized in V is *transportable* if for any other double cone V_1, there is a *-endomorphism ρ_1 localized in V_1 and a unitary operator $U \in \mathcal{A}$ such that $U\rho(A)U^* = \rho_1(A)$ for all $A \in \mathcal{A}$.

     Endomorphisms are better to study since they have more intrinsic structure (product, sometimes inverse) than representations. From the category Δ the unobservable field algebras, the gauge groups, the superselection sectors, and the particle statistics can be nicely reconstructed.

     Inversely, one can also start with a field algebra $\mathcal{F}$ and a gauge group G acting on a Hilbert space and define the observables as the gauge invariant elements of $\mathcal{F}$. One can show that the representation of $\mathcal{F}$ will provide just the DHR representations.

7 Bell inequalities in algebraic quantum field theory

Abstract: The main results will be listed concerning the Bell inequalities in algebraic quantum field theory.

Literature: Halvorson, 2007; Summers and Werner, 1987a,b, 1988; Hofer-Szabó and Vecsernyés, 2013a,b.

- **Bell inequality**. Let $\mathcal{A}$ and $\mathcal{B}$ be two mutually commuting C^*-subalgebras of some C^*-algebra $\mathcal{C}$. A *Bell operator* R for the pair $(\mathcal{A}, \mathcal{B})$ is an element of the following set:

$$\mathbb{B}(\mathcal{A},\mathcal{B}) := \left\{ \frac{1}{2}\big(X_1(Y_1+Y_2) + X_2(Y_1-Y_2)\big) \,\big|\, X_i = X_i^* \in \mathcal{A};\; Y_i = Y_i^* \in \mathcal{B};\; -\mathbf{1} \leqslant X_i, Y_i \leqslant \mathbf{1} \right\}$$

where $\mathbf{1}$ is the unit element of $\mathcal{C}$. For any Bell operator R the following can be proven: (i) For any state $\phi\colon \mathcal{C} \to \mathbb{C}$, one has $|\phi(R)| \leqslant \sqrt{2}$; (ii) for separable states (i.e. for convex combinations of product states) $|\phi(R)| \leqslant 1$.

The *Bell correlation coefficient* of a state ϕ is defined as

$$\beta(\phi,\mathcal{A},\mathcal{B}) := \sup\left\{ |\phi(R)| \,\big|\, R \in \mathbb{B}(\mathcal{A},\mathcal{B}) \right\}$$

and the *Bell inequality* is said to be *violated* if $\beta(\phi,\mathcal{A},\mathcal{B}) > 1$, and *maximally violated* if $\beta(\phi,\mathcal{A},\mathcal{B}) = \sqrt{2}$. Some theorems:

Theorem 1. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then there are some states violating the Bell inequality for $\mathcal{A} \otimes \mathcal{B}$ iff both $\mathcal{A}$ and $\mathcal{B}$ are non-abelian (Bacciagaluppi, 1994).

Theorem 2. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be von Neumann algebras, and suppose that $\mathcal{N}_1$ is abelian and $\mathcal{N}_1 \subseteq \mathcal{N}_2'$ ($\mathcal{N}'$ being the commutant of $\mathcal{N}$). Then for any state $\beta(\phi,\mathcal{N}_1,\mathcal{N}_2) \leqslant 1$. On the other hand, if both $\mathcal{N}_1$ and $\mathcal{N}_2$ are non-abelian von Neumann algebras such that $\mathcal{N}_1 \subseteq \mathcal{N}_2'$, and if $(\mathcal{N}_1,\mathcal{N}_2)$ satisfies the *Schlieder-property*,[8] then there is a state ϕ for which $\beta(\phi,\mathcal{N}_1,\mathcal{N}_2) = \sqrt{2}$ (Landau, 1987)

Theorem 3. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are *properly infinite*[9] von Neumann algebras on the Hilbert space $\mathcal{H}$ such that $\mathcal{N}_1 \subseteq \mathcal{N}_2'$, and $(\mathcal{N}_1,\mathcal{N}_2)$ satisfies the Schlieder-property, then there is a dense set of vectors in $\mathcal{H}$ inducing states which violate the Bell inequality across $(\mathcal{N}_1,\mathcal{N}_2)$ (Halvorson and Clifton, 2000).

Theorem 4. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{R}$ be a von Neumann factor of type III_1 acting on $\mathcal{H}$. Then every normal state ϕ of $\mathcal{B}(\mathcal{H})$ maximally violates the Bell inequality across $(\mathcal{R},\mathcal{R}')$ (Summers and Werner, 1988).

Theorem 5. The vacuum state maximally violates the Bell inequality across the *wedge*[10] algebras $(\mathcal{N}(W),\mathcal{N}(W)')$. (Summers, Werner 1988).

[8] The commuting pair $(\mathcal{A},\mathcal{B})$ of C^*-subalgebras in $\mathcal{C}$ obeys the Schlieder-property, if for $0 \neq A \in \mathcal{A}$ and $0 \neq B \in \mathcal{B}$, $AB \neq 0$. Since in case of von Neumann algebras A and B can be required to be projections, Schlieder-property is the analogue of logical independence in classical logic.

[9] The center contains no finite projections.

[10] Poincaré transforms of the region $W_R := \{x \in \mathcal{M} | x_1 > |x_0|\}$.

- **Bell inequality in AQFT**. The above theorems will hold in AQFT. Since the local von Neumann algebras supported in spacelike separated double cones satisfy the Schlieder property, therefore Theorem 2 applies to these algebras stating that there is a state maximally violating the Bell inequality across these local algebras. Similarly, Theorem 3 applies to local von Neumann algebras supported in spacelike separated double cones stating that there is a dense set of vectors in $\mathcal{H}$ inducing states which violate the Bell inequality. Finally, the local von Neumann algebras supported in spacelike separeted double cones satisfy the assumptions of Theorem 4, therefore every normal state will maximally violate the Bell inequality across pairs of algebras supported in spacelike separated double cones.

- **The CHSH inequality**. The Bell inequality typically used in AQFT is of the following form:

$$\big|\phi\big(X_1(Y_1+Y_2)+X_1(Y_1-Y_2)\big)\big| \leqslant 2, \tag{45}$$

where $X_m \in \mathcal{N}(V_A)$ and $Y_n \in \mathcal{N}(V_B)$ are self-adjoint *contractions* (that is $-\mathbf{1} \leqslant X_m, Y_n \leqslant \mathbf{1}$ for $m,n = 1,2$) supported in spatially separated spacetime regions V_A and V_B, respectively. This type of Bell inequality is usually referred to as the *Clauser–Horne–Shimony–Holte (CHSH) inequality.*

- **The CH inequality**. Sometimes in the EPR-Bell literature another Bell-type inequality is used: the *Clauser–Horne (CH) inequality* defined in the following way:

$$-1 \leqslant \phi(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) \leqslant 0, \tag{46}$$

where A_m and B_n are *projections* located in $\mathcal{N}(V_A)$ and $\mathcal{N}(V_B)$, respectively. It is easy to see, however, that the two inequalities are equivalent: in a given state ϕ the set $\{(A_m, B_n); m,n = 1,2\}$ violates the CH inequality (46) *if and only if* the set $\{(X_m, Y_n); m,n = 1,2\}$ of self-adjoint contractions given by

$$X_m := 2A_m - \mathbf{1} \tag{47}$$

$$Y_n := 2B_n - \mathbf{1} \tag{48}$$

violates the CHSH inequality (45).

8 Locality and causality concepts in local physical theories

Abstract: The most important notions of locality and causality will be listed in the framework of a local physical theory.

Literature: Haag, 1992; Hofer-Szabó and Vecsernyés, 2015a,b; Earman and Valente, 2014

- **Von Neumann algebras as local algebras**. Consider a classical field theory on $\mathcal{M}$ with configuration space $F^{\mathcal{M}} := \{\Phi\colon \mathcal{M} \to F\}$ with field values $F = \mathbb{R}^n, \mathbb{C}^n$. The maximal σ-algebra of classical events would be the power set $\mathcal{P}(F^{\mathcal{M}})$ of the set of field configurations, but this is not consistent with the net structure. However, by taking the equivalence classes of those field configurations which have the same field values on a given spacetime region, $\Phi \sim_V \Psi$ if $\Phi_{|V} = \Psi_{|V}$, one can generate local cylindrical σ-algebras.

  The hard and unsolved problem is to give a probability measure on the σ-algebra $(F^{\mathcal{M}}, \mathcal{P}(F^{\mathcal{M}}))$ or on a meaningful σ-subalgebra of it. We can avoid this conundrum by choosing a locally finite covering of $\mathcal{M}$, and restricting the field configurations to be piecewise constant on regions corresponding to minimal elements in the covering. We can simplify further the situation by restricting the field values F to a finite set.

  Note that the projections of a local von Neumann algebra do not possess a direct spacetime localization: they project to subsets of $F^{\mathcal{M}}$ and not to those of $\mathcal{M}$.

- **Operations**. A linear map $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ is said to be *positive* if $\mathcal{T}(A^*A) \geqslant 0$ for any $A \in \mathcal{A}$. $\mathcal{T}$ is said to be *completely positive* if its linear extension on elementary tensors:

  $$\mathcal{T} \otimes \mathrm{id}_n : \mathcal{A} \otimes M_n \to A \otimes M_n; \quad (\mathcal{T} \otimes \mathrm{id}_n)(A \otimes B) = \mathcal{T}(A) \otimes B \tag{49}$$

  is positive for any $n \in \mathbb{N}$. Completely positive maps with $\mathcal{T}(\mathbf{1}) \leqslant \mathbf{1}$ are called *(quantum) operations*. If $\mathcal{T}(\mathbf{1}) = \mathbf{1}$, the operation is called *non-selective*; if $\mathcal{T}(\mathbf{1}) < \mathbf{1}$ it is called *selective*. A non-selective operation $\mathcal{T}$ defines an (affine) mapping $\mathcal{T}^*$ of the state space by $\phi \mapsto \phi' = \phi \circ \mathcal{T}$.

  On a type I factor $\mathcal{T}$ is a non-selective operation iff it is *inner*, that is it has a Kraus representation $\mathcal{T} := \sum_i \mathrm{Ad}\, K_i$ where all K_i are positive and $\sum_i K_i = \mathbf{1}$. If the K_i-s are mutually orthogonal projections, then $\mathcal{T}$ is called a *projective* operation. A special projective operation is the Lüders projection:

  $$\mathcal{T}_P(A) := PAP + (\mathbf{1} - P)A(\mathbf{1} - P) \tag{50}$$

  If $\mathcal{A}$ is a von Neumann algebra one usually requires T to be normal.

- **No-signalling**. Let $\{A_k\}_{k\in K} \subset \mathcal{N}(V_A)$ be a partition of the unit (a set of mutually orthogonal projections in the local von Neumann algebra $\mathcal{N}(V_A)$ such that $\sum_k A_k = \mathbf{1}$). Define the corresponding *non-selective* projective measurement (even a conditional expectation) as:

$$\mathcal{T}_{\{A_k\}}\colon \mathcal{A}_{\mathcal{H}} \to \mathcal{A}_{\mathcal{H}}; \quad \mathcal{T}_{\{A_k\}}(X) := \sum_{k\in K} A_k X A_k, \quad X \in \mathcal{A}_{\mathcal{H}} \tag{51}$$

which maps states to states via

$$\phi \mapsto \phi_{\{A_k\}} := \phi \circ \mathcal{T}_{\{A_k\}} \tag{52}$$

Now, *no-signalling* is a local causality principle stating that projections (quantum events) located in spatially separated regions should be insensitive of such a change of states: Let $V_A, V_B \in \mathcal{K}$ be spacelike separated. For any partition $\{A_k\}_{k\in K} \subset \mathcal{N}(V_A)$ and projection $B \in \mathcal{N}(V_B)$, and for any locally faithful and normal state $\phi\colon \mathcal{A}_{\mathcal{H}} \to \mathbb{C}$, we have

$$\phi_{\{A_k\}}(B) = \phi(B) \tag{53}$$

No-signaling follows from microcausality. Schlieder (1969) showed that the converse also holds: if no-signaling holds for a decomposition of the unit $\{A_k\}_{k\in K}$ and a projection B for all normal states of a von Neumann algebra, then $[A_k, B] = 0$ for all $k \in K$. Being equivalent to microcausality no-signaling trivially fulfils in LCTs. Although it is formulated as a requirement for states, it gives a restriction for the structure of the local algebras.

- **Independence**. Instead of non-selective projective measurements (51) one can also consider *selective projective measurements* using a single local projection $A \in \mathcal{N}(A)$:

$$\mathcal{T}_A\colon \mathcal{A}_{\mathcal{H}} \to \mathcal{A}_{\mathcal{H}}; \quad \mathcal{T}_A(X) := AXA \tag{54}$$

which defines a completely positive but not unit preserving map:

$$\phi \mapsto \phi_A := \frac{\phi \circ \mathcal{T}_A}{\phi(A)} = \frac{\phi \circ \mathcal{T}_A}{(\phi \circ \mathcal{T}_A)(\mathbf{1})} \tag{55}$$

called *Lüders projection*. If defined on the whole $\mathcal{B}(\mathcal{H})$, Lüders projected state ϕ_A is the unique normal state on $\mathcal{B}(\mathcal{H})$ with the property that for any projection $B \leqslant A \in \mathcal{B}(\mathcal{H})$, $\phi_A(B) = \phi(B)/\phi(A)$.

Now, *independence* is the following local causality requirement: For any projections $A \in \mathcal{N}(V_A)$ and $B \in \mathcal{N}(V_B)$ such that $V_A, V_B \in \mathcal{K}$ are spacelike separated regions, and for any locally faithful and normal state ϕ, we have[11]

$$\phi_A(B) = \phi(B) \tag{56}$$

[11] Butterfield (1995, Eqs. 3.6 and 3.7) and Earman and Valente (2014, Sec. 7.2) call (53) and (56) *parameter independence* and *outcome independence*, respectively (Shimony, 1986). For the difference between parameter independence, where ϕ in (53) is *conditioned* on the common cause, and no-signaling, where ϕ is *unconditioned*, see (Maudlin 2002) and (Norsen 2011).

In case of microcausality (56) implies that $\phi(AB) = \phi(A)\phi(B)$, that is ϕ becomes a product state by restricting it to the subalgebra generated by $\mathcal{N}(V_A)$ and $\mathcal{N}(V_B)$. Hence, it is a too strong assumption, which is violated in LQTs, for example, by any entangled state. Of course, it is violated also in case of superluminal correlations.

- **Local primitive causality.** For any globally hyperbolic bounded subspacetime region $V \in \mathcal{K}$, $\mathcal{A}(V'') = \mathcal{A}(V)$.

- **Local determinism**. A net satisfying local primitive causality also satisfies *local determinism* (Earman and Valente, 2014): For any two states ϕ and ϕ' and for any globally hyperbolic spacetime region $V \in \mathcal{K}$, if $\phi|_{\mathcal{A}(V)} = \phi'|_{\mathcal{A}(V)}$ then $\phi|_{\mathcal{A}(V'')} = \phi'|_{\mathcal{A}(V'')}$ and consequently it also satisfies

- **Stochastic Einstein locality**: Let $V_A, V_C \in \mathcal{K}$ such that $V_C \subset J_-(V_A)$ and $V_A \subset V_C''$. If $\phi|_{\mathcal{A}(V_C)} = \phi'|_{\mathcal{A}(V_C)}$ holds for any two states ϕ and ϕ' on $\mathcal{A}$ then $\phi(A) = \phi'(A)$ for any projection $A \in \mathcal{A}(V_A)$.

- **Entailments**. If a net satisfies Haag duality, then it also satisfies local primitive causality. But microcausality alone does not entail local primitive causality. Since microcausality is equivalent to no-signaling and local primitive causality represents no-superluminal propagation (Earman and Valente, 2014), therefore it is an interesting question whether there exist nets which *satisfy* local primitive causality but *violate* microcausality. Field algebras serve such examples: Although local field algebras are defined to be relatively local to observables

$$\mathcal{F}(V) := \mathcal{A}(V')' \cap \mathcal{F}, \quad V \in \mathcal{K}, \tag{57}$$

  local field algebras corresponding to spacelike separated regions do not commute in general, hence microcausality fails. However, local primitive causality does hold in the net of field algebras, because $V' = V'''$ and hence

$$\mathcal{F}(V) := \mathcal{A}(V')' \cap \mathcal{F} = \mathcal{A}(V''')' \cap \mathcal{F} = \mathcal{A}((V'')')' \cap \mathcal{F} =: \mathcal{F}(V''), \quad V \in \mathcal{K}. \tag{58}$$

  Thus, for such a net of local (field) algebras no-signaling is violated whereas no-superluminal propagation holds.

- **Primitive causality** is a global version of local primitive causality (entailed by it): Let $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{K}$ be a covering collection of a Cauchy surface $\mathcal{C}$ and let $\mathcal{A}(\mathcal{K}(\mathcal{C}))$ be the corresponding algebra. Then $\mathcal{A}(\mathcal{K}(\mathcal{C})) = \mathcal{A}$.

- **Determinism**. If $\phi|_{\mathcal{A}(\mathcal{K}_\mathcal{C})} = \phi'|_{\mathcal{A}(\mathcal{K}_\mathcal{C})}$ for any two states ϕ and ϕ' on $\mathcal{A}$ then $\phi = \phi'$. A local physical theory with primitive causality satisfies determinism.

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