Bell inequality and common causal explanation in algebraic quantum field theory

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Project

- **Question:** What is the relation between
  - the Bell inequalities
  - and the common causal explanation of correlations

in algebraic quantum field theory (AQFT)?

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  Classical case: Common cause $\Longrightarrow$ Bell inequality
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- $$\begin{array}{lccc} \textbf{Classical case:} & \text{Common cause} & \Longrightarrow & \text{Bell inequality} \\ & & & \downarrow \\ \textbf{Quantum case:} & & & \text{Bell inequality} \end{array}$$

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  in algebraic quantum field theory (AQFT)?

- $$\begin{array}{llcl} \textbf{Classical case:} & \text{Common cause} & \Longrightarrow & \text{Bell inequality} \\ & \quad\;\;\color{red}{\downarrow} & & \quad\;\;\downarrow \\ \textbf{Quantum case:} & \quad\;\;\color{red}{?} & \color{red}{\Longrightarrow} & \text{Bell inequality} \end{array}$$

Project

I. Classical common causal explanation
II. Nonclassical common causal explanation
III. One correlation: Common Cause Principles in AQFT
IV. More correlations: joint common causal explanation in AQFT

Reichenbachian common cause

- **Classical probability space:** (Σ, p)
- **Positive correlation:** A, $B \in \Sigma$

$$p(AB) > p(A)\, p(B)$$

- **Reichenbachian common cause:** $C \in \Sigma$

$$\begin{aligned} p(AB|C) &= p(A|C)p(B|C) \\ p(AB|\overline{C}) &= p(A|\overline{C})p(B|\overline{C}) \\ p(A|C) &> p(A|\overline{C}) \\ p(B|C) &> p(B|\overline{C}) \end{aligned}$$

Common cause system

- **Correlation:** A, $B \in \Sigma$

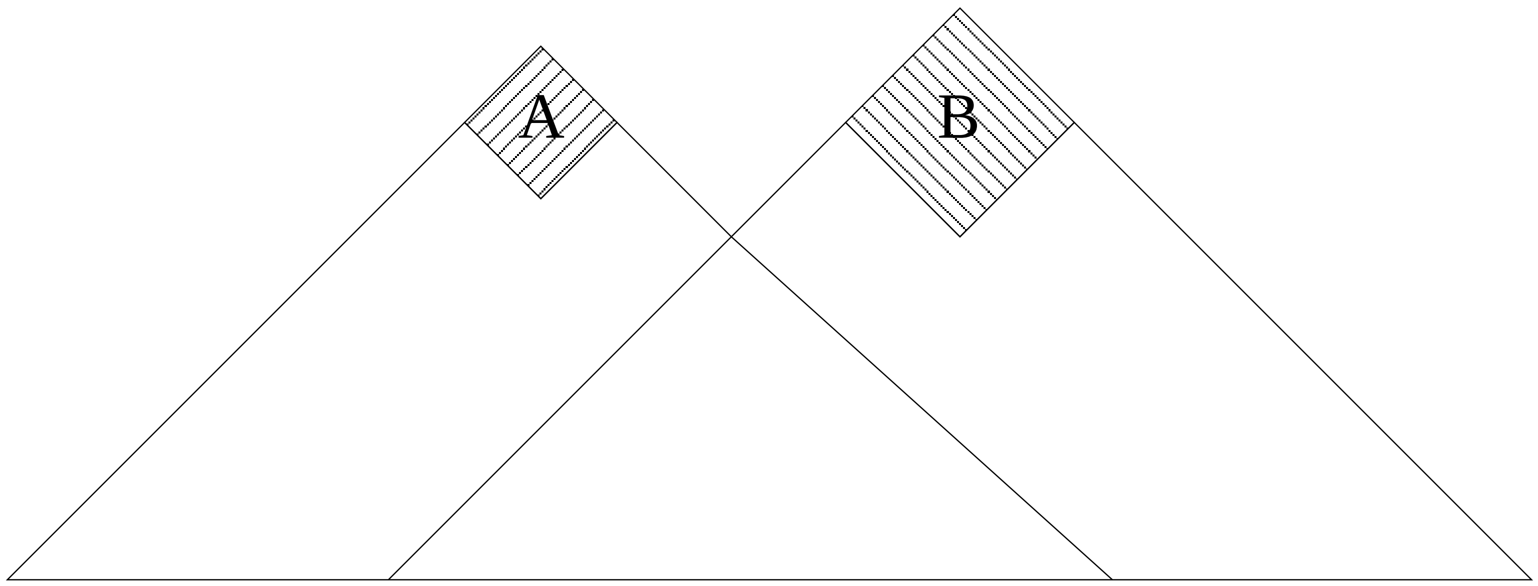
$$p(AB) \neq p(A)p(B)$$

- **Common cause system** (CCS): partition $\{C_k\}_{k \in K}$ in Σ

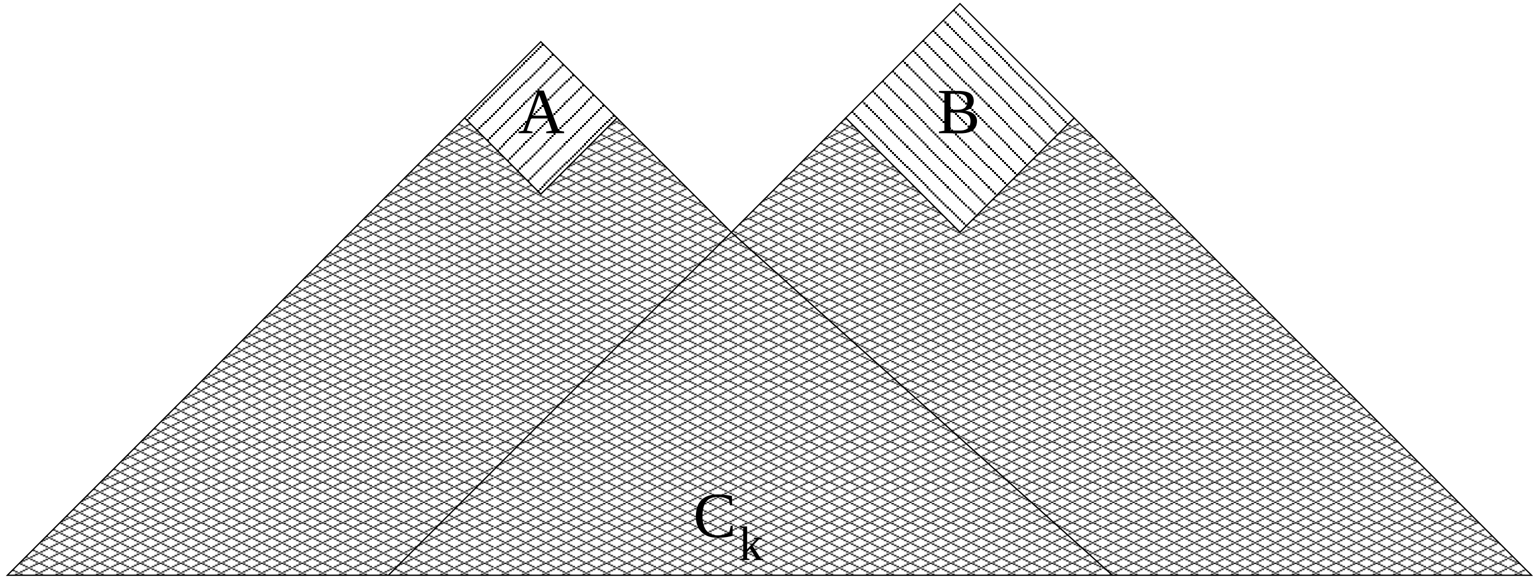
$$p(AB|C_k) \;=\; p(A|C_k)\, p(B|C_k)$$

- **Common cause:** CCS of size 2.

Localization of the common cause

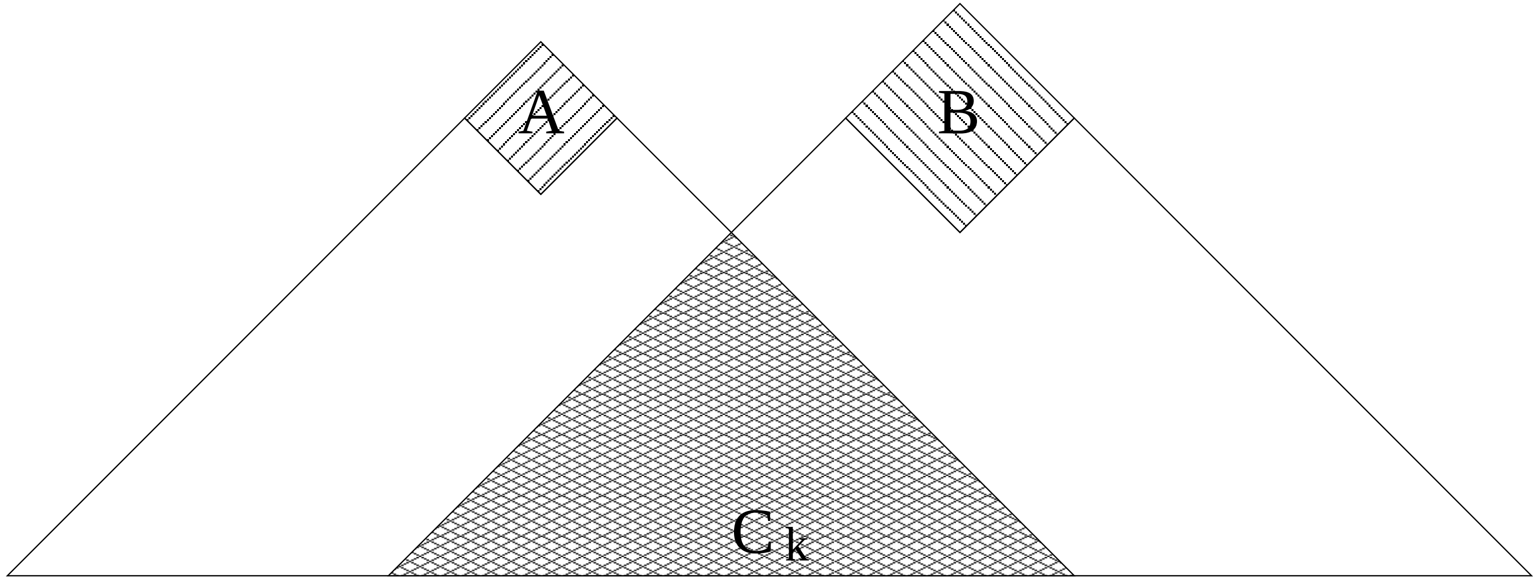


Localization of the common cause



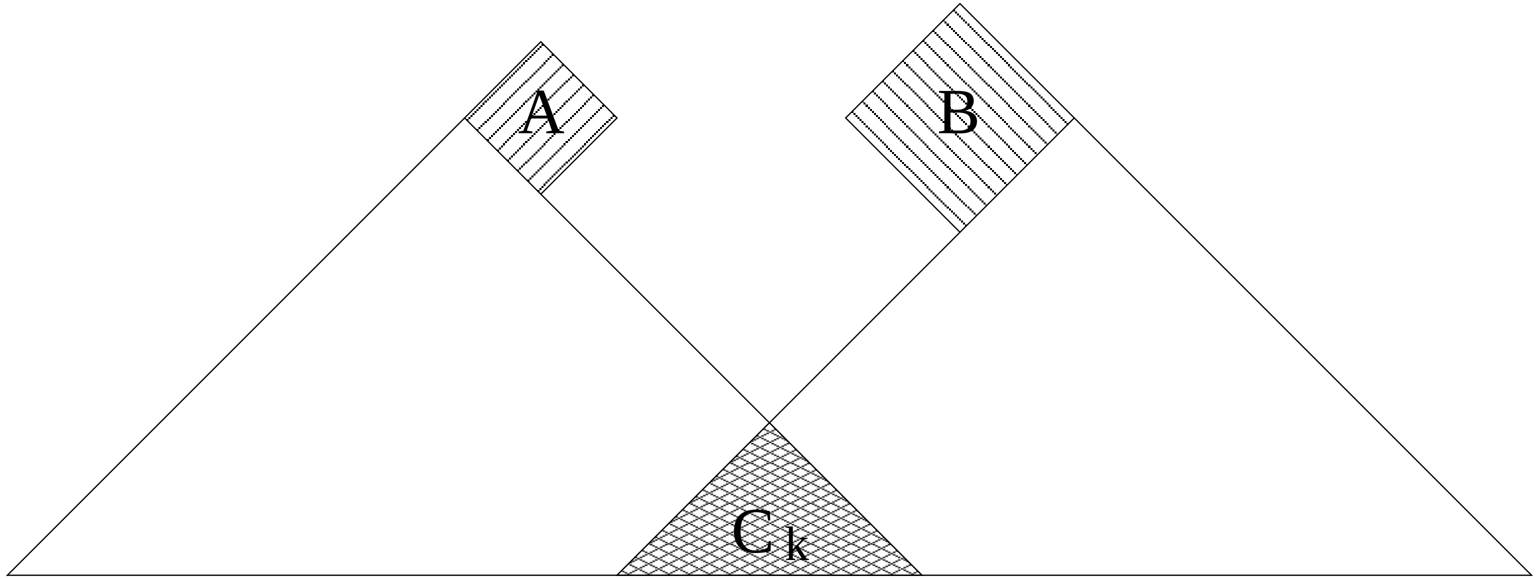
- Weak common cause system

Localization of the common cause



- Common cause system

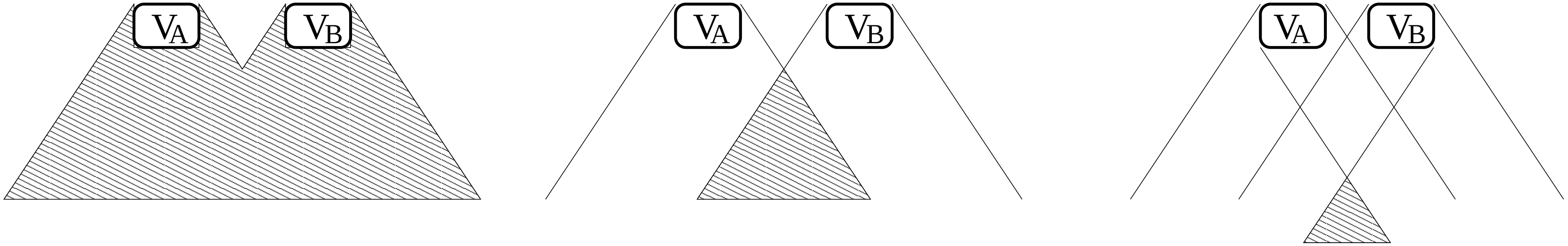
Localization of the common cause



- Strong common cause system

Localization of the common cause

- V_A and V_B: localization of A and B.



Weak past: $wpast(V_A, V_B) := I_-(V_A) \cup I_-(V_B)$

Common past: $cpast(V_A, V_B) := I_-(V_A) \cap I_-(V_B)$

Strong past: $spast(V_A, V_B) := \cap_{x \in V_A \cup V_B} I_-(x)$

Joint common cause system

- **Correlations:** A_m, $B_n \in \Sigma$ $(m \in M, n \in N)$

$$p(A_m B_n) \neq p(A_m)\, p(B_n)$$

- **Joint CCS:** partition $\{C_k\}_{k \in K}$ in Σ

$$p(A_m B_n | C_k) \;\; = \;\; p(A_m | C_k)\, p(B_n | C_k)$$

Conditional probabilities

- **Measurement outcomes:** $A_m,\ B_n \in \Sigma \ (m \in M, n \in N)$
- **Measurement choices:** $a_m,\ b_n \in \Sigma \ (m \in M, n \in N)$ also localized in V_A and V_B
- **Conditional correlations:**

$$p(A_m B_n | a_m b_n) \neq p(A_m | a_m)\, p(B_n | b_n)$$

Common causal explanation

- **Local, non-conspriratorial joint common causal explanation**: a partition $\{C_k\}$ in Σ

$$p(A_m B_n|a_m b_n C_k) = p(A_m|a_m b_n C_k)\, p(B_n|a_m b_n C_k) \qquad \text{(screening-off)}$$

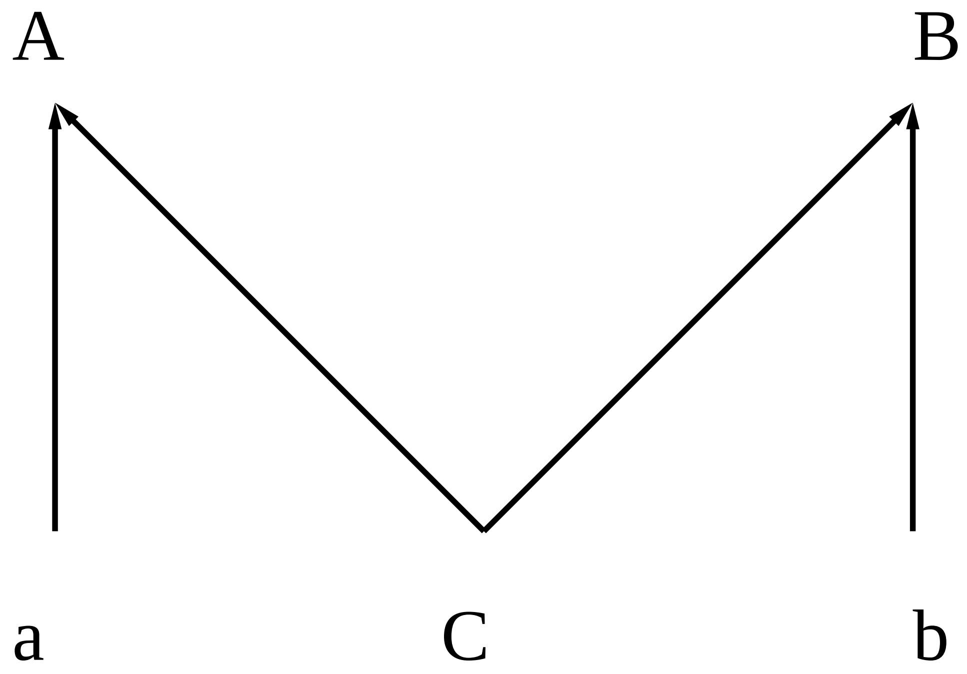
$$p(A_m|a_m b_n C_k) = p(A_m|a_m C_k) \qquad \text{(locality)}$$

$$p(B_n|a_m b_n C_k) = p(B_n|b_n C_k) \qquad \text{(locality)}$$

$$p(a_m b_n C_k) = p(a_m b_n)\, p(C_k) \qquad \text{(no-conspiracy)}$$

- $\{C_k\}$ can be localized in any of the three different pasts

Motivation by Markov condition



- **Markov condition** $\Longrightarrow$ screening-off, locality and no-conspiracy

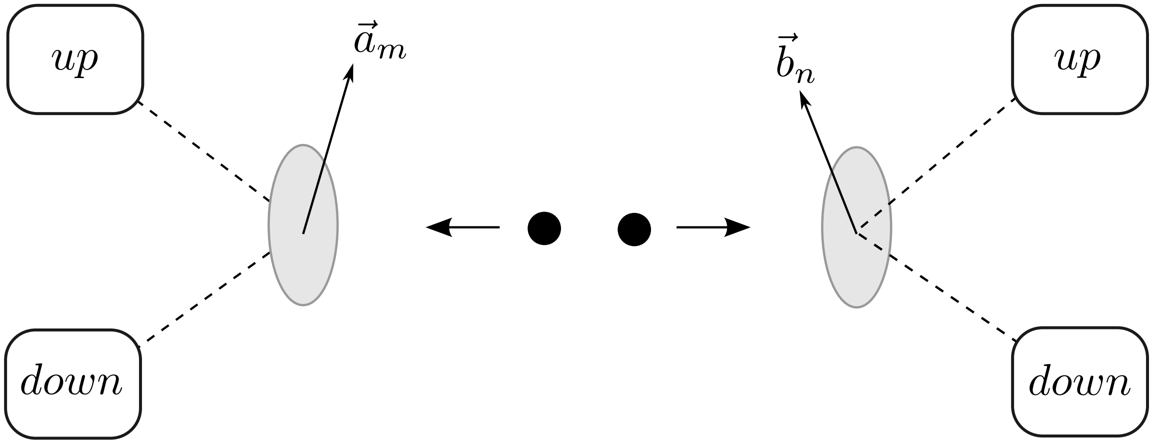
Clauser–Horne inequality

$$\begin{array}{r} \text{Joint CCS} \\ \text{Locality} \\ \text{No-conspiracy} \end{array} \quad \Longrightarrow \quad \text{CH inequality}$$

$$\begin{aligned} -1 \leqslant p(A_1B_1|a_1b_1) + p(A_1B_2|a_1b_2) + p(A_2B_1|a_2b_1) \\ -p(A_2B_2|a_2b_2) - p(A_1|a_1) - p(B_1|b_1) \leqslant 0 \end{aligned}$$

(which is equivalent to the CHSH inequality.)

EPR correlations



- Conditional probabilities:

$$p(A_m|a_m),\ p(B_n|b_n),\ p(A_mB_n|a_mb_n) \qquad (m,n=1,2)$$

- CH inequalitity is violated.
- Therefore: no common causal explanation for EPR.

Classical ontology

- ”Bell inequalities are relations between conditional probabilities valid under the locality assumption.” (Gisin, 2009)

Quantum ontology

- **Event space:** von Neumann lattice
- **Events:** projections
- **Probability:** quantum state
- **CH inequality:**

$$-1 \leqslant \phi\,(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) \leqslant 0$$

Quantum ontology

- **Event space:** von Neumann lattice
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- *Then let us take the quantum ontology seriously!*
- **Common cause:**
  1. What is that?
  2. How it relates to the CH inequality?

Non-classical common cause system

- **Non-classical probability space:** $(\mathcal{P}(\mathcal{N}), \phi)$
- **Correlation:** A, $B \in \mathcal{P}(\mathcal{N})$

$$\phi(AB) \neq \phi(A)\phi(B)$$

- **(Non-classical) CCS:** partition $\{C_k\}_{k \in K}$ in $\mathcal{P}(\mathcal{N})$

$$\frac{\phi(C_k ABC_k)}{\phi(C_k)} = \frac{\phi(C_k AC_k)}{\phi(C_k)} \frac{\phi(C_k BC_k)}{\phi(C_k)}$$

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- **Commuting / Noncommuting CCS:** $\{C_k\}_{k\in K}$ is commuting / not commuting with A and B
- **Nontrivial CCS**: $C_k \nleq A, A^\perp, B$ or $B^\perp$ for some $k \in K$

Joint common cause system

- **Set of correlations:** A_m, $B_n \in \mathcal{P}(\mathcal{N})$

$$\phi(A_m B_n) \neq \phi(A_m)\phi(B_n)$$

- **Joint CCS:** partition $\{C_k\}_{k\in K}$ in $\mathcal{P}(\mathcal{N})$

$$\frac{\phi(C_k A_m B_n C_k)}{\phi(C_k)} = \frac{\phi(C_k A_m C_k)}{\phi(C_k)} \frac{\phi(C_k B_n C_k)}{\phi(C_k)}$$

- **A subtle point:**

  Joint CCS = local, non-conspiratorial joint CCS

Clauser–Horne inequality

- *Commuting* joint CCS (Locality) (No-conspiracy) $\Longrightarrow$ CH inequality

$$-1 \leqslant \phi\left(A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 - A_1 - B_1\right) \leqslant 0$$

- EPR has no *commutative* common causal explanation

Noncommuting common causes

- But why to demand commutativity between a cause and its effects?
- Standard QM: operators *do not commute* with their time translates:
  - Harmonic oscillator: $x(t) \equiv U(t)^{-1} x U(t)$

$$\left[x(t),\, x\right]\psi_0 = -\frac{i\hbar}{m\omega}\sin\left(\hbar\omega t\right)\psi_0 \not\equiv 0$$

Noncommuting common causes

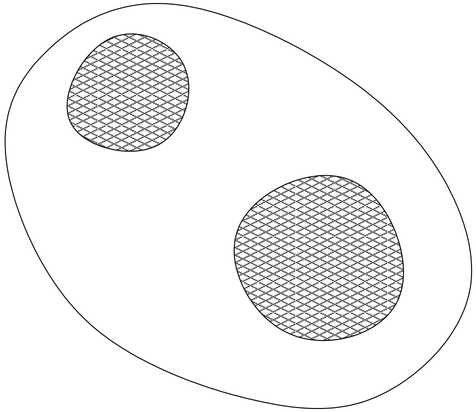
- $$\left.\begin{array}{r}\textit{Noncommuting}\text{ joint CCS}\\ \text{(Locality)}\\ \text{(No-conspiracy)}\end{array}\right. \quad \not\Rightarrow \quad \text{CH inequality}$$
- **Question:** Can a *set* of correlations violating the CH inequality have a noncommuting *joint* common causal explanation in AQFT?

Noncommuting common causes

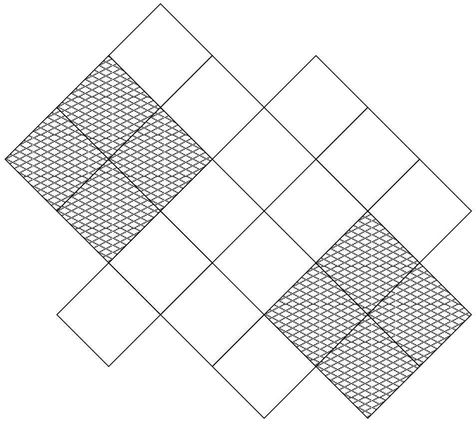
- **An easier question:** Can *one* correlation have a common causal explanation in AQFT? (Rédei 1997)
- **Common Cause Principle** (CCP): If there is a correlation between two events and there is no direct causal (or logical) connection between them, then there always exists a common cause of the correlation.

Algebraic quantum field theory

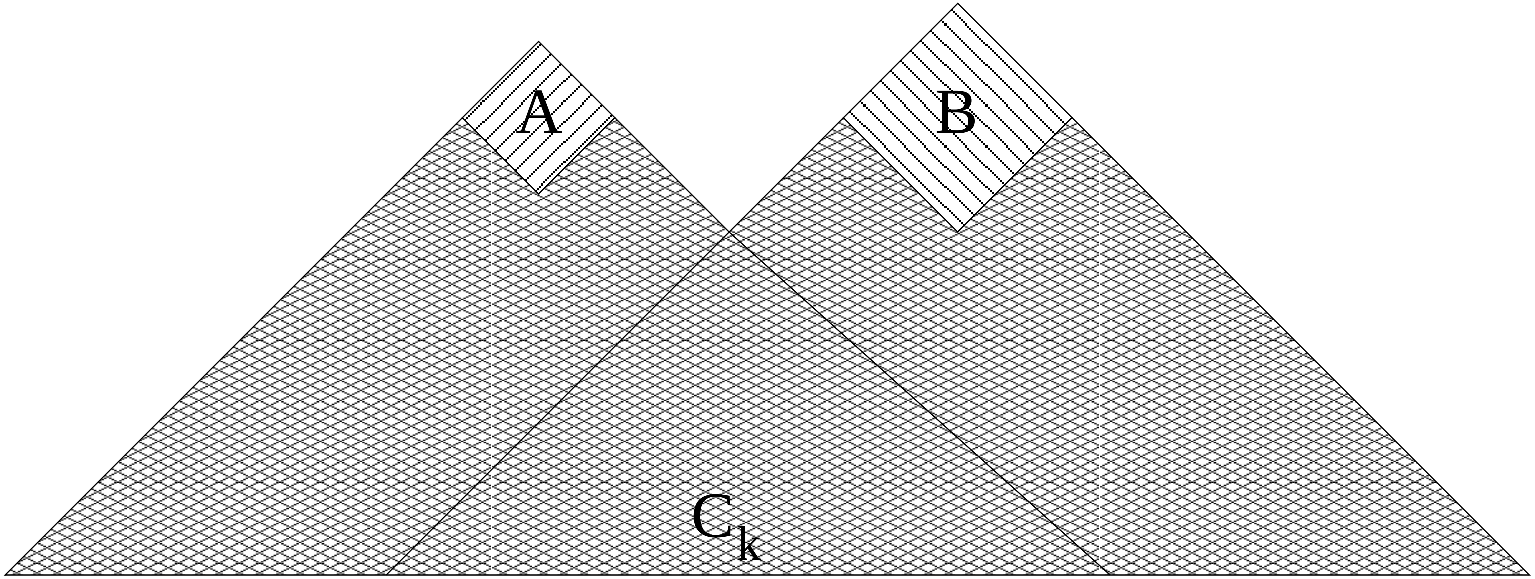
- **Poincaré covariant AQFT:**



- **Quantum Ising model:**

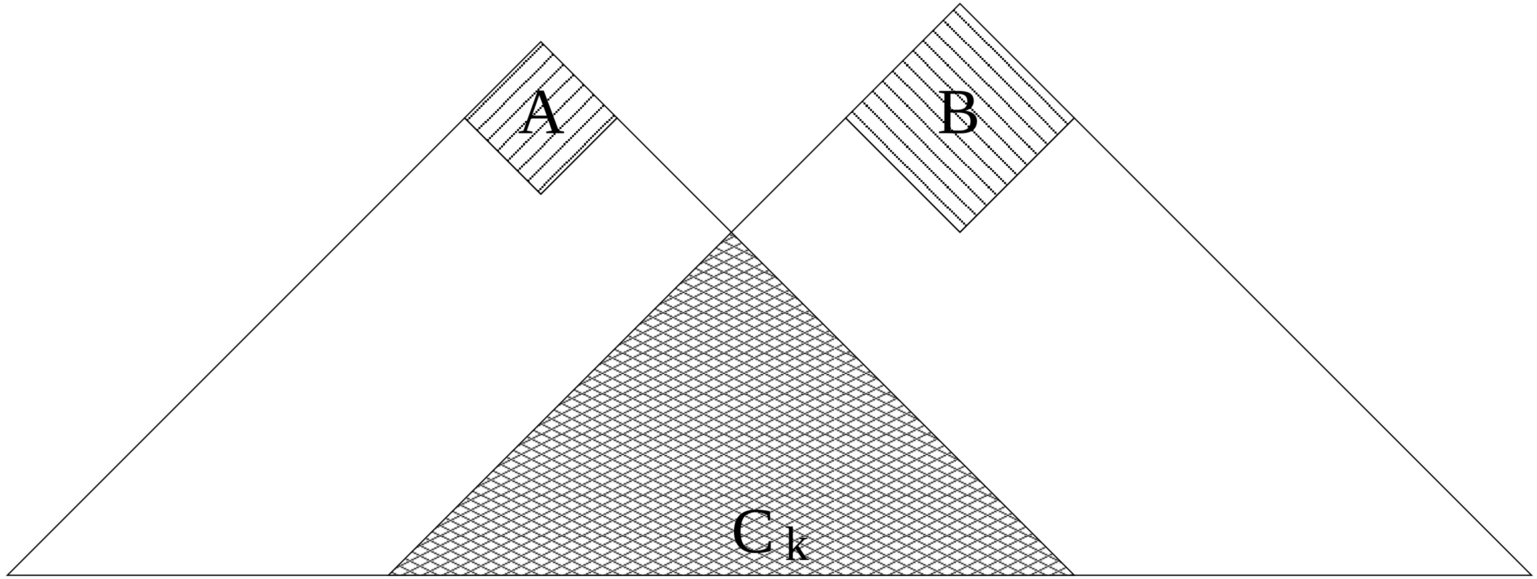


Common Cause Principles in AQFT



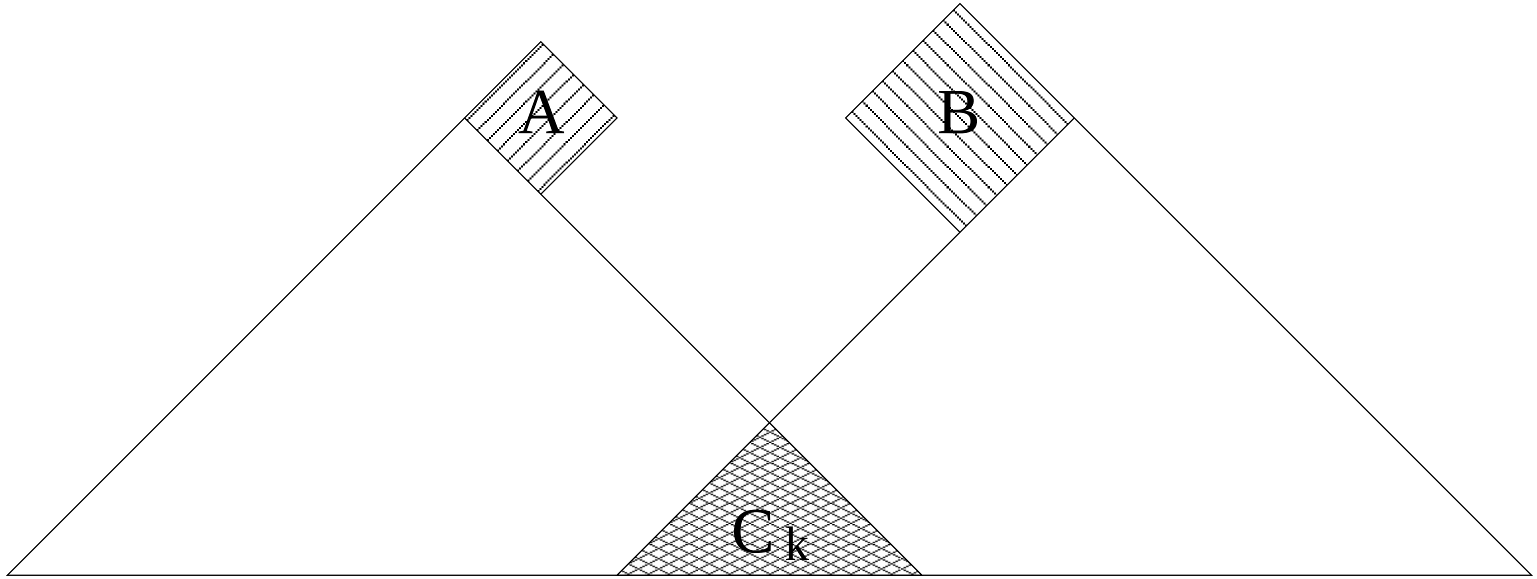
- Weak (Commutative/Noncommutative) CCP

Common Cause Principles in AQFT



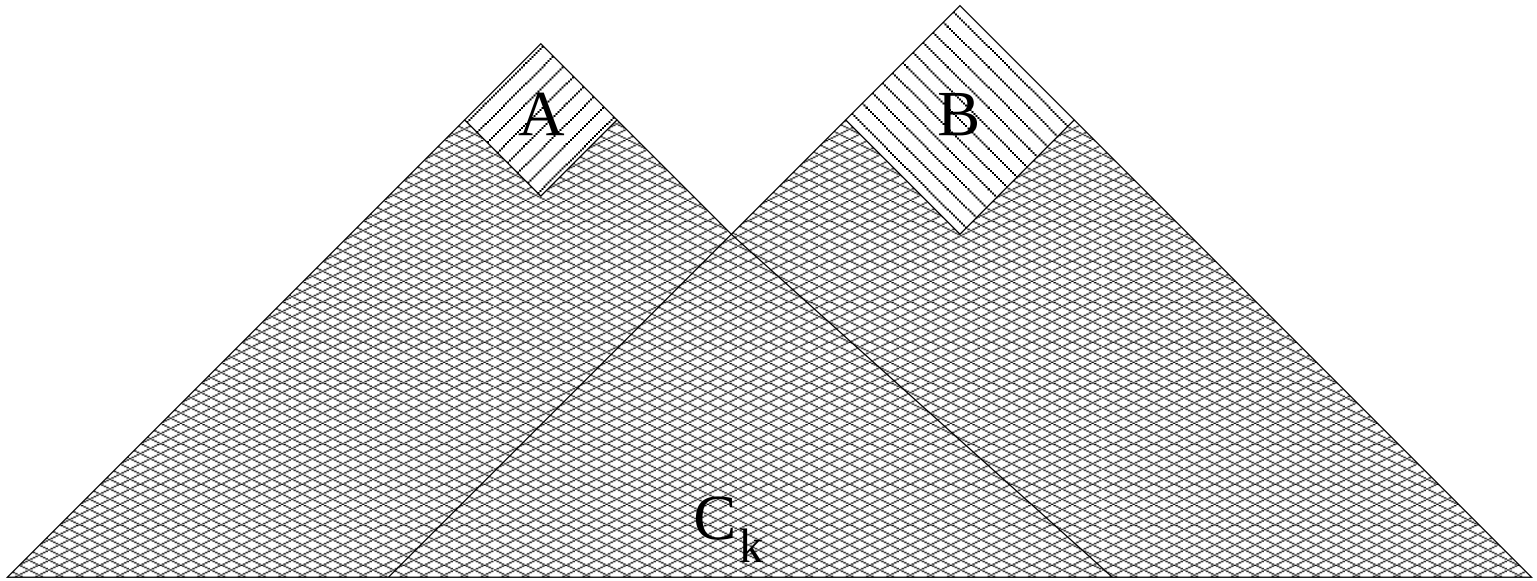
- (Commutative/Noncommutative) CCP

Common Cause Principles in AQFT



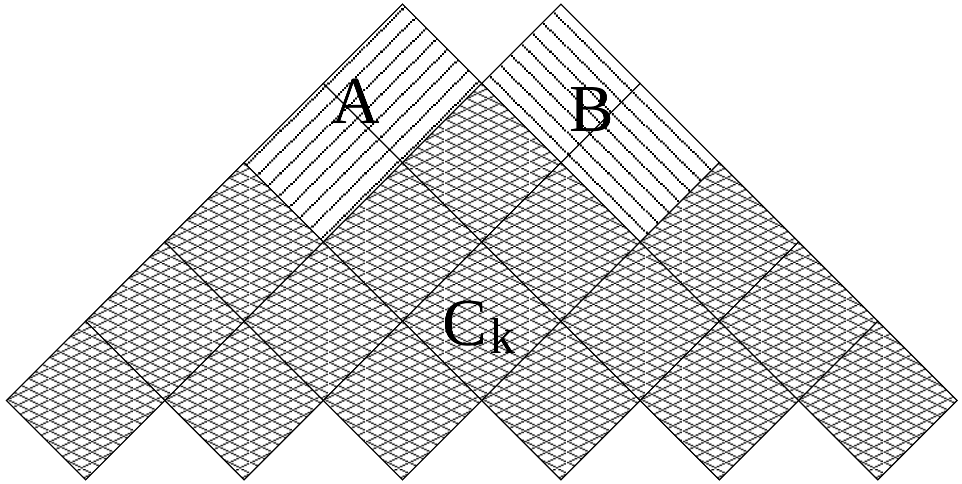
- Strong (Commutative/Noncommutative) CCP

Common Cause Principles in AQFT



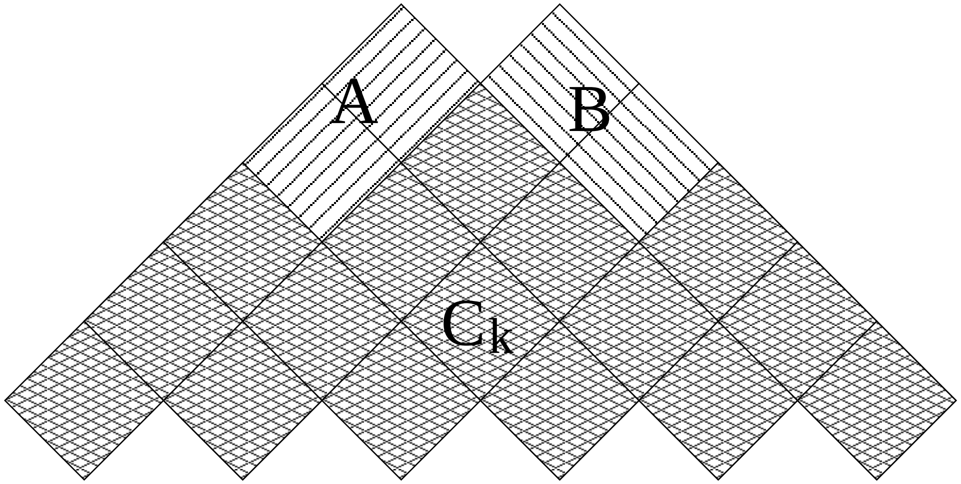
- **Proposition:** The Weak Commutative CCP *holds* in Poincaré covariant AQFT (Rédei, Summers, 2002).
- **Question:** What about other AQFTs?

Common Cause Principles in AQFT



- **Proposition:** The Weak Commutative CCP *does not* hold in quantum Ising model (Hofer-Szabó, Vecsernyés, 2012a).
- **Question:** What about abandoning commutativity?

Common Cause Principles in AQFT

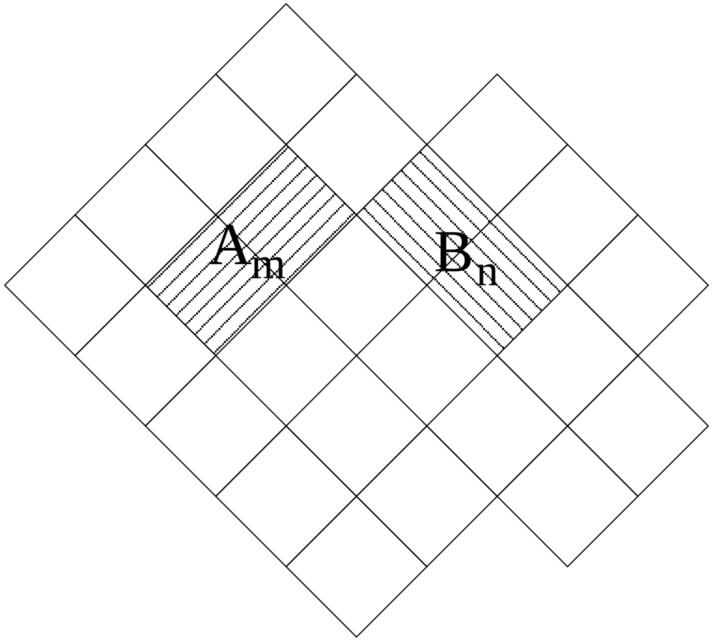


- **Proposition:** The Weak Noncommutative CCP *holds* in quantum Ising model (Hofer-Szabó, Vecsernyés, 2012b).

Joint common cause system in AQFT

- **Original question:** Can a *set* of correlations violating the CH inequality have a noncommuting *joint* common causal explanation in AQFT?

Correlations violating CH



- $A_m = A(\overrightarrow{\mathbf{a}}^m)$, $B_n = B(\overrightarrow{\mathbf{b}}^n)$: four projections $(m, n = 1, 2)$
- $\overrightarrow{\mathbf{a}}^m$, $\overrightarrow{\mathbf{b}}^n$: Bell directions
- ρ^s: singlet state
- Maximal violation of the CH (CHSH) inequality.

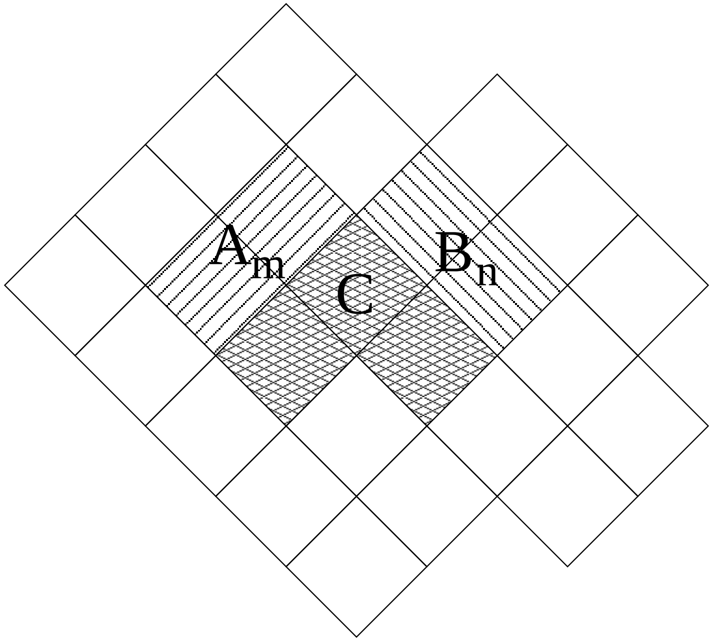
Noncommuting common causes

- ... after some calculation ...

$$
\begin{aligned}
\rho_C &= \mathbf{1} + \lambda U_{-\frac{1}{2}} U_{\frac{1}{2}} \\
&+ \frac{1+\lambda}{2} c_1 (U_{-\frac{1}{2}} + U_{\frac{1}{2}}) + \frac{1-\lambda}{2} c_1' (U_{-\frac{1}{2}} - U_{\frac{1}{2}}) \\
&+ \frac{1+\lambda}{2} c_2 (U_0 - U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) - \lambda c_2 (U_{-1} U_0 U_1 + U_{-1} U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}} U_1) \\
&+ \frac{1-\lambda}{2} c_2' (U_0 + U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) \\
&+ \frac{1+\lambda}{2} c_3 i (U_{-\frac{1}{2}} U_0 - U_0 U_{\frac{1}{2}}) + \frac{1-\lambda}{2} c_3' i (U_{-\frac{1}{2}} U_0 + U_0 U_{\frac{1}{2}}) \\
&+ \lambda c_1 c_2 (U_{-1} U_{-\frac{1}{2}} U_0 U_1 + U_{-1} U_0 U_{\frac{1}{2}} U_1) \\
&+ \lambda c_2^2 (-U_{-1} U_1 + U_{-1} U_{-\frac{1}{2}} U_{\frac{1}{2}} U_1) \\
&+ \lambda c_2 c_3 i (U_{-1} U_{-\frac{1}{2}} U_1 - U_{-1} U_{\frac{1}{2}} U_1).
\end{aligned}
$$

- **Answer:** Yes.

Joint common cause system in AQFT



- **Proposition:** (Hofer-Szabó, Vecsernyés, 2012c) There is a *noncommuting* common cause $\{C, C^{\perp}\}$ of the correlations $\{(A_m, B_n)\}$; and it can be localized in the shaded region.

Conclusion

- **Classical case:** Common cause $\Longrightarrow$ Bell inequality

$$\downarrow$$

Quantum case: Bell inequality

Conclusion

- $$\begin{array}{llcl} \textbf{Classical case:} & \text{Common cause} & \Longrightarrow & \text{Bell inequality} \\ & \quad\quad\downarrow & & \quad\quad\downarrow \\ \textbf{Quantum case:} & \text{Common cause} & \not\Longrightarrow & \text{Bell inequality} \end{array}$$
- The violation of the Bell inequality in AQFT does *not* exclude a set of correlations to have a joint common causal explanation *if* commutativity is abandoned.

Remarks

- In the noncommutative case the theorem of total probability does *not* hold. (No ’Hempelian’ explanation.)
- Are the (Strong/Weak) Noncommutative *Joint* CCPs valid in AQFT?
- What are the ontological consequences of applying noncommutative common causes?

References

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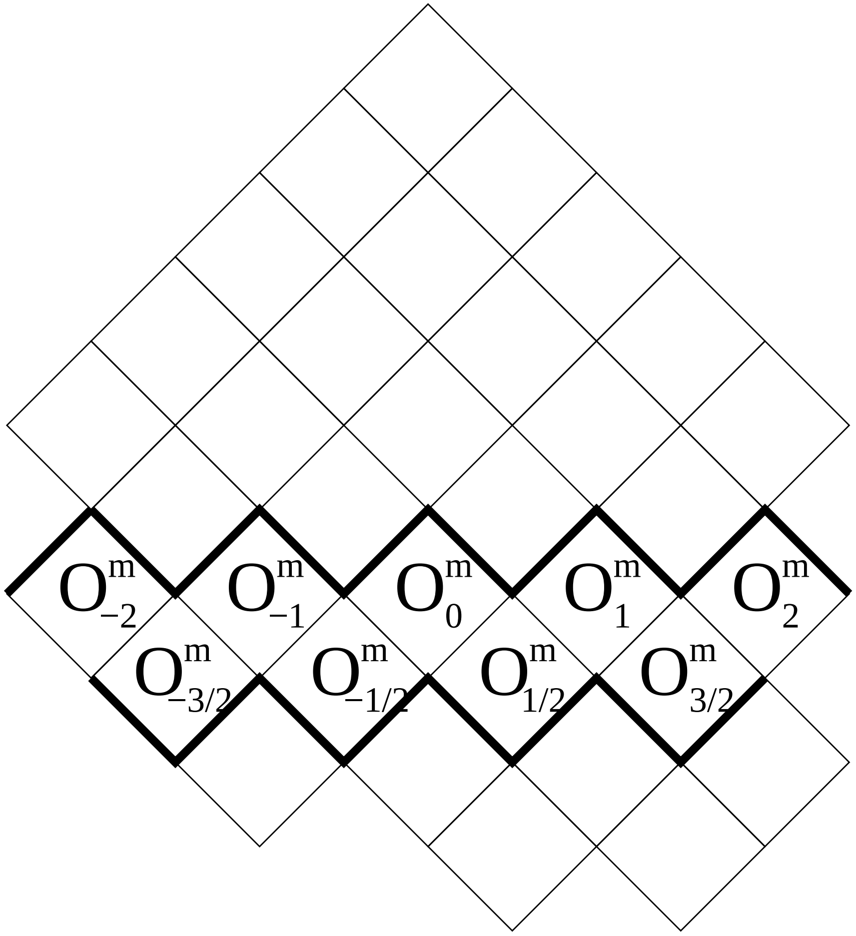
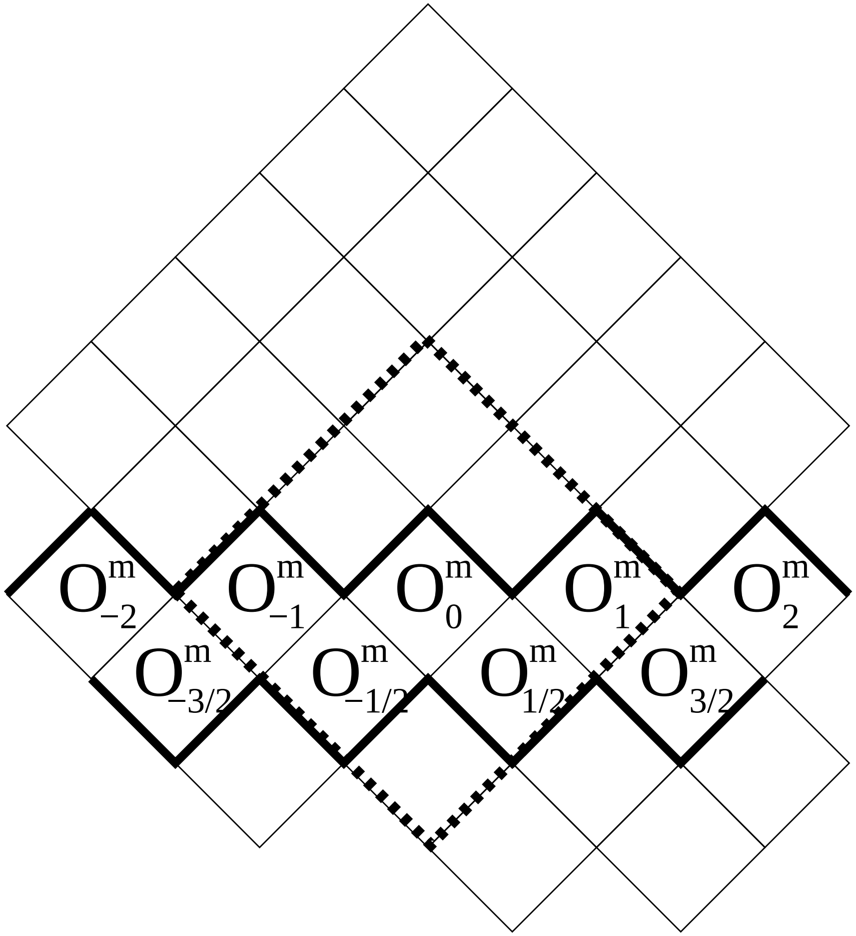
Quantum Ising model



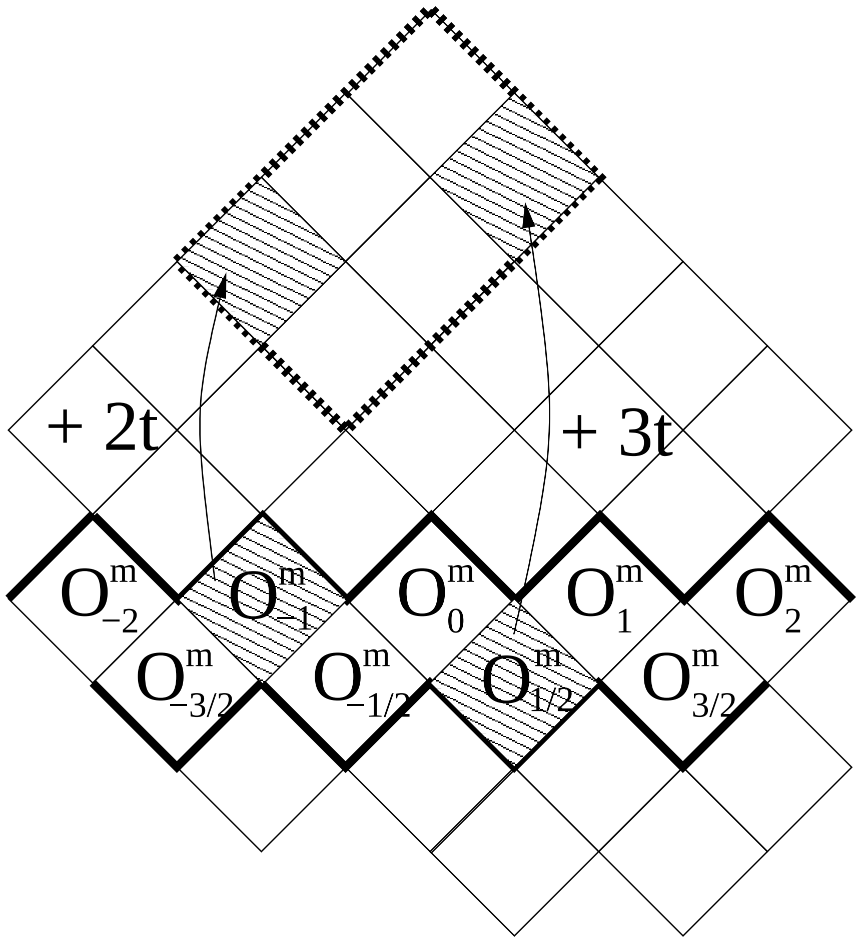
- **Minimal double cones:** $\mathcal{O}_i^m$

Quantum Ising model



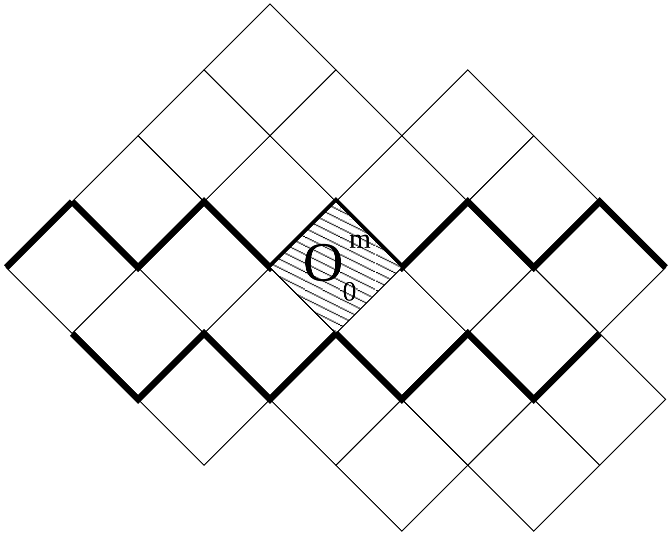
- **Double cones:** $\mathcal{O}_{i,j}$, smallest double cone containing $\mathcal{O}^m_i$ and $\mathcal{O}^m_j$

Quantum Ising model



- **Net:** $\mathcal{K}^m$, by integer time translation

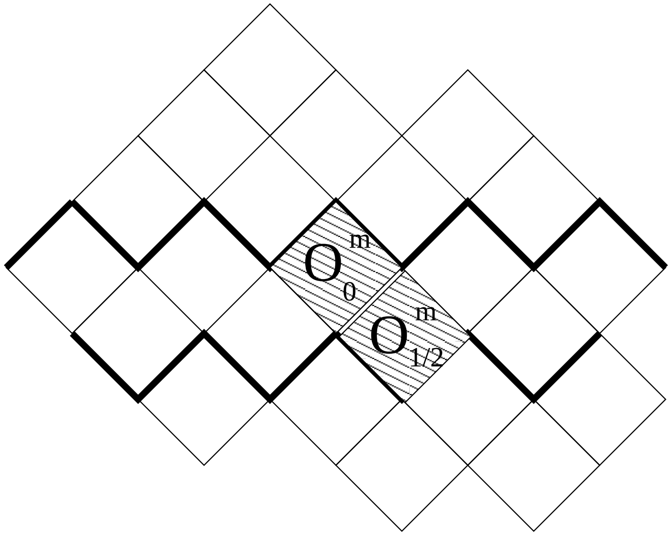
‘One-point’ algebras



- **Linear basis:** $1,\ U_0$
- **Minimal projections:** $P = \frac{1}{2}\left(1 \pm U_0\right)$
- **Commutation relations:**

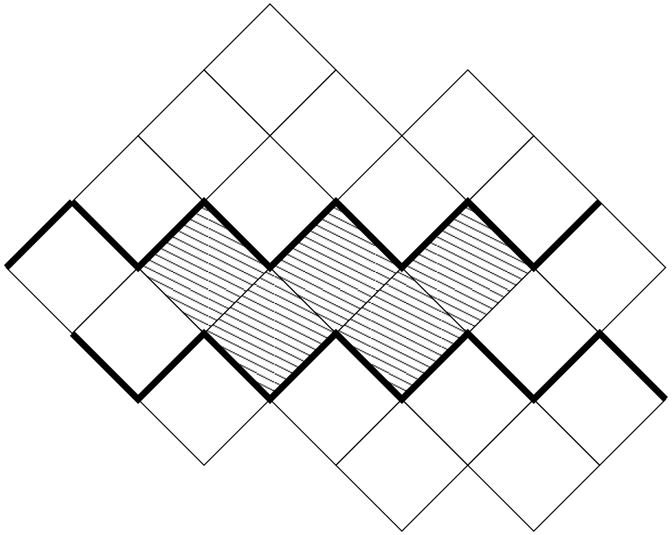
$$U_i U_j = \begin{cases} -U_j U_i, & \text{if } |i-j| = \frac{1}{2} \\ U_j U_i, & \text{otherwise} \end{cases}$$

‘Two-point’ algebras



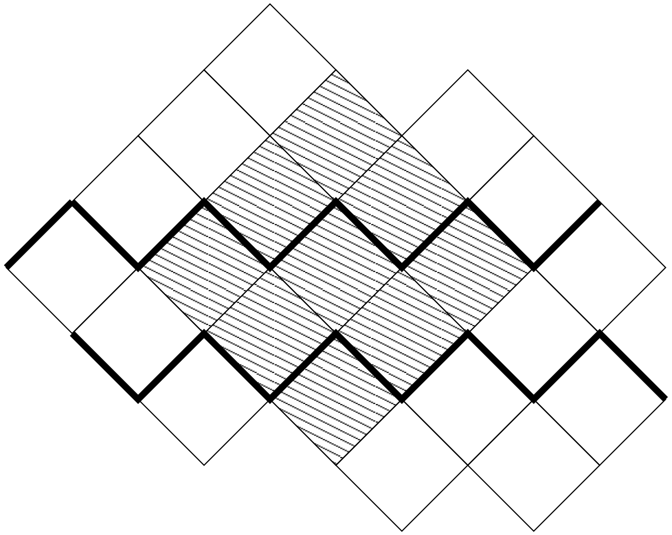
- **Linear basis:** $\mathbf{1}, U_0, U_{\frac{1}{2}}, iU_0U_{\frac{1}{2}}$
- **Minimal projections:** $P = \frac{1}{2}\left(\mathbf{1} + \overrightarrow{\mathbf{n}} \cdot \mathbf{U}\right), \quad \overrightarrow{\mathbf{n}} \in \mathbf{R}^3$

Dynamics



- **Dynamics:** automorphisms of $\mathcal{A}$ (Müller, Vecsernyés 2012)
- Local primitive causality holds.

Dynamics



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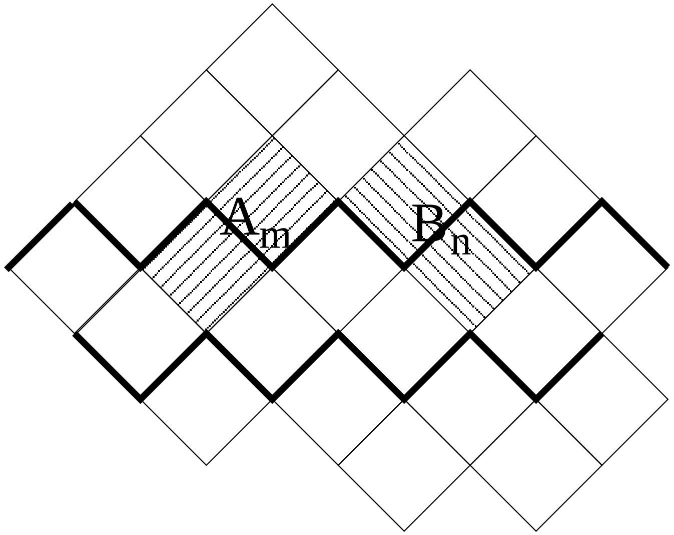
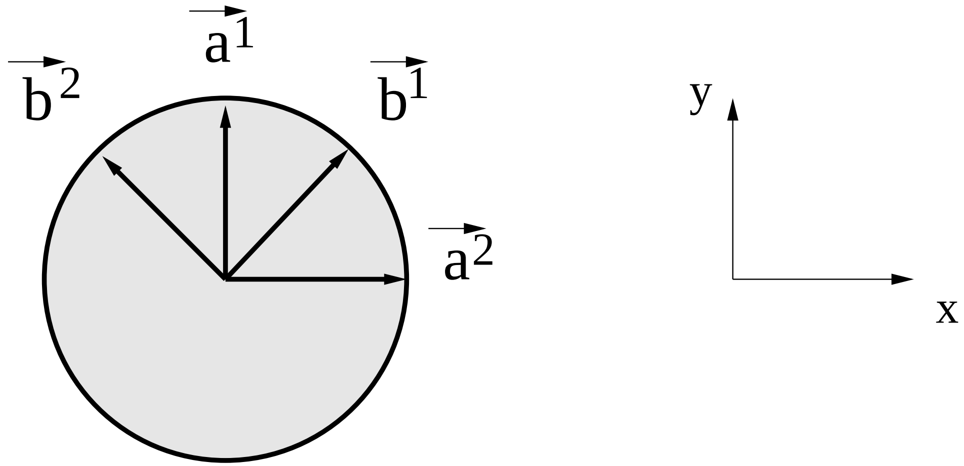
Correlations violating CH



- $A_m = A(\overrightarrow{\mathbf{a}}^m)$, $B_n = B(\overrightarrow{\mathbf{b}}^n)$: four projections $(m, n = 1, 2)$
- ρ^s: singlet state

Correlations violating CH

- Directions:



maximally violating of the CH inequality ...

Correlations violating CHSH

- ... or, equivalently, the CHSH inequality:

$$\left|\phi\Big(U_1(V_1+V_2)+U_2(V_1-V_2)\Big)\right| \quad \leqslant \quad 2$$

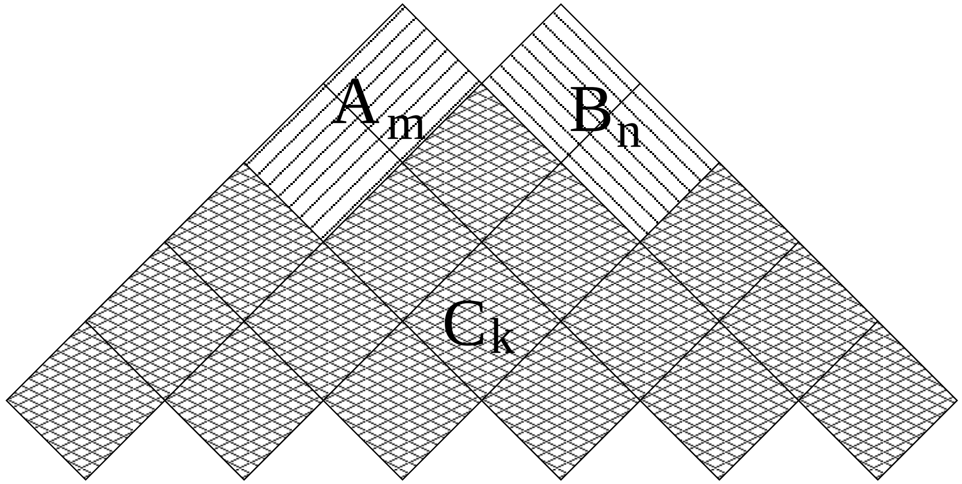
where

$$\begin{aligned} U_m &:= 2A_m - \mathbf{1} \\ V_n &:= 2B_n - \mathbf{1} \end{aligned}$$

Correlations violating CHSH

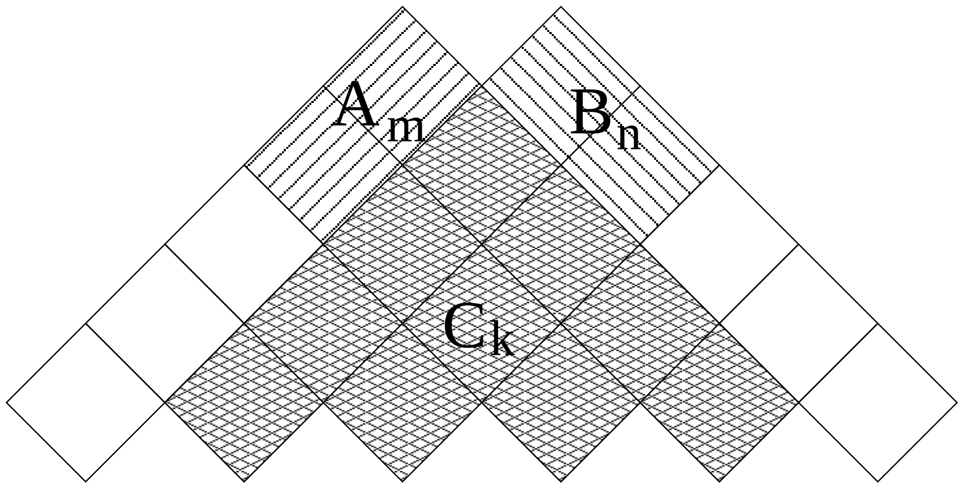
- **Question:** Can these four correlations have a noncommutative joint common causal explanation?

Localization of the common cause



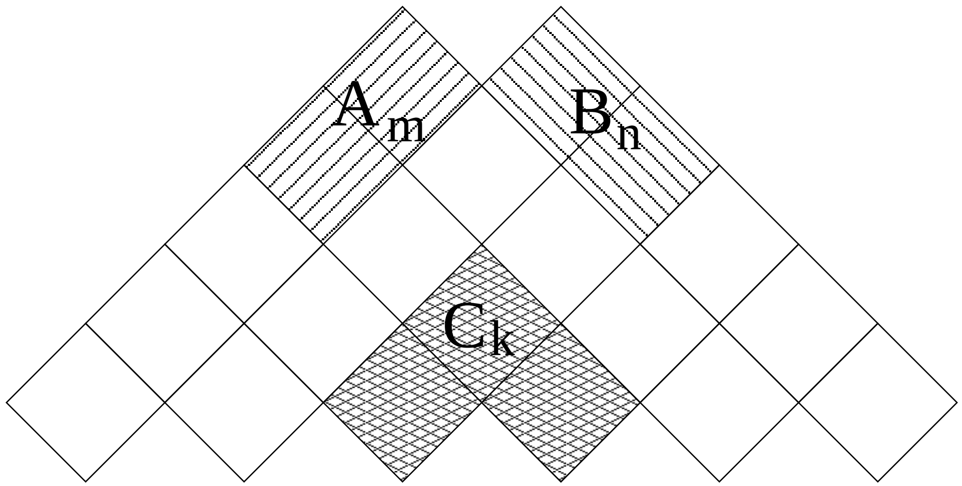
- Weak joint common cause system

Localization of the common cause



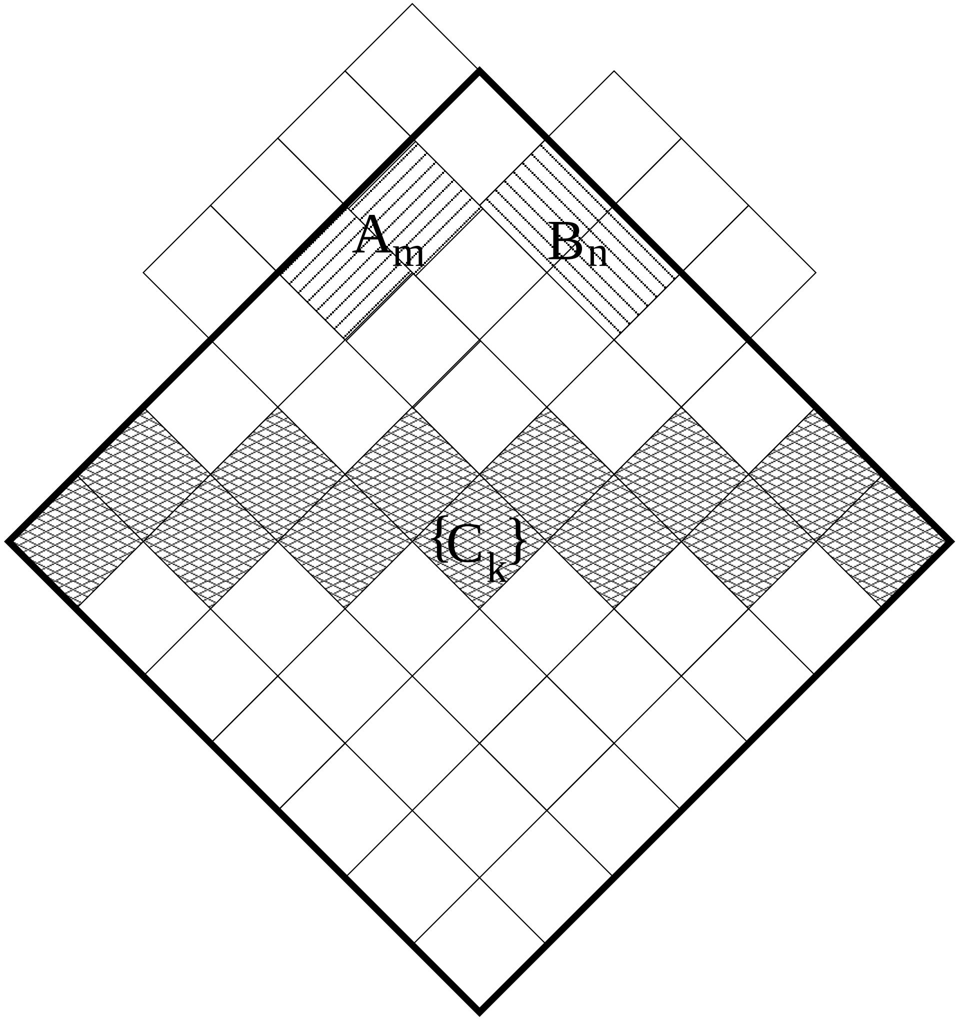
- Joint common cause system

Localization of the common cause



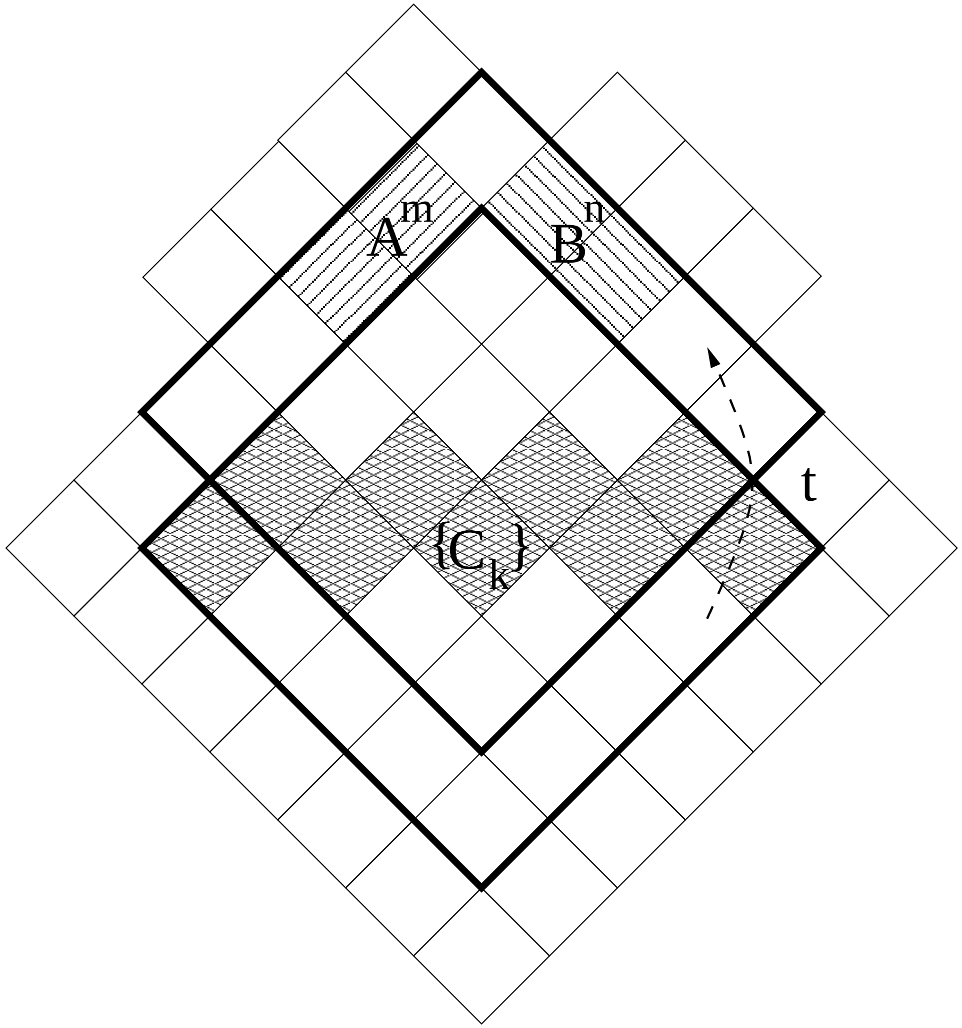
- Strong joint common cause system

Localization of the common cause



- Weak joint common cause system: one needs only local primitivity and isotony (no dynamics)

Localization of the common cause



- (Strong) joint common cause system: one needs also dynamics

Bell inequality in AQFT

- $\mathcal{A}$ and $\mathcal{B}$: two mutually commuting C^*-subalgebras of C
- **Bell operator** for $(\mathcal{A}, \mathcal{B})$: R, an element of the set

$$\mathbb{B}(\mathcal{A}, \mathcal{B}) \equiv \Big\{ \frac{1}{2}\big(A_1(B_1 + B_2) + A_1(B_1 - B_2)\big) \;\Big|\; A_i = A_i^* \in \mathcal{A};\ B_i = B_i^* \in \mathcal{B};\ -\mathbf{1} \leqslant A_i, B_i \leqslant \mathbf{1} \Big\}$$

Bell inequality in AQFT

- **Bell correlation coefficient** of a state ϕ:

$$\beta(\phi, \mathcal{A}, \mathcal{B}) \equiv \sup\left\{ |\phi(R)| \;\middle|\; R \in \mathbb{B}(\mathcal{A}, \mathcal{B}) \right\}$$

- The **Bell inequality** is *violated* if

$$|\beta(\phi, \mathcal{A}, \mathcal{B})| > 1$$

Mathematical results

- **Proposition:** If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras then there are some states violating the Bell inequality for $\mathcal{A} \otimes \mathcal{B}$ iff both $\mathcal{A}$ and $\mathcal{B}$ are non-abelian (Bacciagaluppi, 1994).
  - Going over to von Neumann algebras ... (Landau 1987)
  - Adding further constraints ... (Summer-Werner, 1988; Halvorson, Clifton, 2000)
  - The above theorems apply in ”typical” AQFTs ...

Joint common cause system

Joint CCS = local, non-conspiratorial joint CCS

Proof:

- Rewriting both the classical and the non-classical local, non-conspiratorial joint CCS in an *indexical form*.
- 'Translating' quantum probabilities into classical conditional probabilities by the *Kolmogorovian Censorship Hypothesis*.

Non-classical joint common cause system

Correlation:

$$\phi(A_m B_n) \;\neq\; \phi(A_m)\,\phi(B_n)$$

Indexical notation:

$$\phi_{C_k}(X) \;:=\; \frac{(\phi \circ E_c)(XC_k)}{\phi(C_k)} = \frac{\phi(C_k X C_k)}{\phi(C_k)}.$$

Non-classical, local, non-conspiratorial joint CCS:

$$\begin{aligned}
\phi_{C_k}(A_m B_n) &= \phi_{C_k}(A_m)\,\phi_{C_k}(B_n) \\
\phi_{C_k}(A_m) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m B_n^{\perp}) \\
\phi_{C_k}(B_n) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m^{\perp} B_n) \\
\phi_{C_k}(\mathbf{1}) &= 1.
\end{aligned}$$

Kolmogorovian Censorship Hypothesis

Let $(\mathcal{N}, \mathcal{P}(\mathcal{N}), \phi)$ be a non-classical probability space. Let Γ be a countable set of non-commuting selfadjoint operators in $\mathcal{N}$. For every $Q \in \Gamma$, let $\mathcal{P}(Q)$ be a maximal Abelian sublattice of $\mathcal{P}(\mathcal{N})$ containing all the spectral projections of Q. Finally, let a map $p_0 : \Gamma \to [0,1]$ be such that

$$\sum_{Q \in \Gamma} p_0(Q) = 1, \qquad p_0(Q) > 0.$$

Then there exists a classical probability space (Ω, Σ, p) such that for every projection X^Q in any $\mathcal{P}(Q)$ there exist events X_{cl}^Q and x_{cl}^Q in Σ such that

$$\begin{aligned} X_{cl}^Q &\subset x_{cl}^Q \\ x_{cl}^Q \cap x_{cl}^R &= 0, \quad \text{if } Q \neq R \\ p(x_{cl}^Q) &= p_0(Q) \\ \phi(X^Q) &= p(X_{cl}^Q | x_{cl}^Q) \end{aligned}$$

Classical joint common cause system

Correlation:

$$p(A_m \wedge B_n \,|\, a_m \wedge b_n) \quad \neq \quad p(A_m|a_m)\, p(B_n|b_n)$$

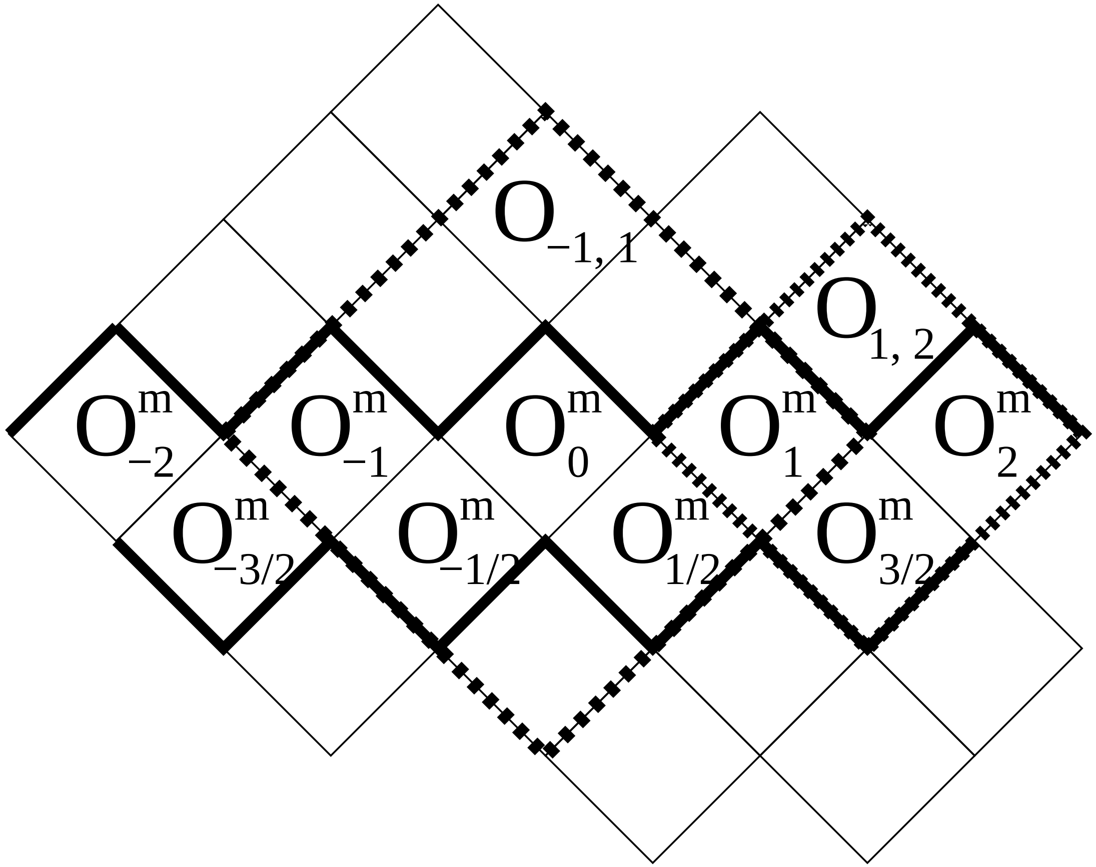
Indexical notation:

$$p_{C_k}(X|x) \quad := \quad \frac{p(X \wedge C_k|x)}{p(C_k)}.$$

Classical, local, non-conspiratorial joint CCS:

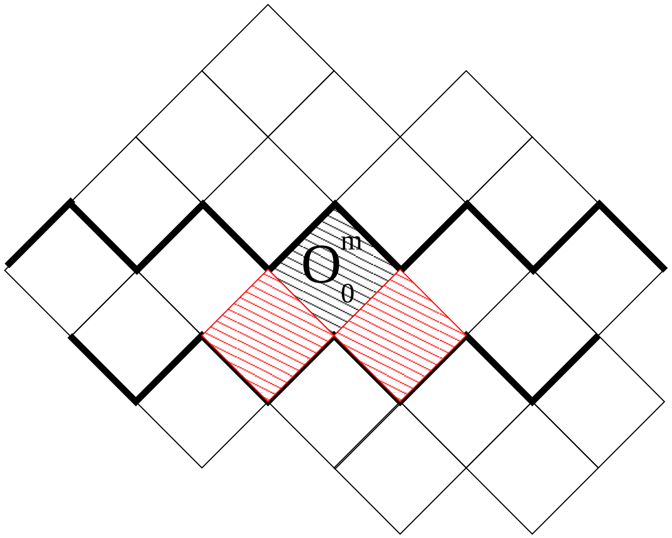
$$\begin{aligned}
p_{C_k}(A_m \wedge B_n|a_m \wedge b_n) &= p_{C_k}(A_m|a_m \wedge b_n)\, p_{C_k}(B_n|a_m \wedge b_n), \\
p_{C_k}(A_m|a_m \wedge b_n) &= p_{C_k}(A_m|a_m \wedge b_{n'}), \\
p_{C_k}(B_n|a_m \wedge b_n) &= p_{C_k}(B_n|a_{m'} \wedge b_n), \\
p_{C_k}(\Omega|a_m \wedge b_n) &= 1.
\end{aligned}$$

Quantum Ising model



- **Cauchy surface net:** $\mathcal{K}^m_{CS}$, poset of double cones based on the Cauchy surface

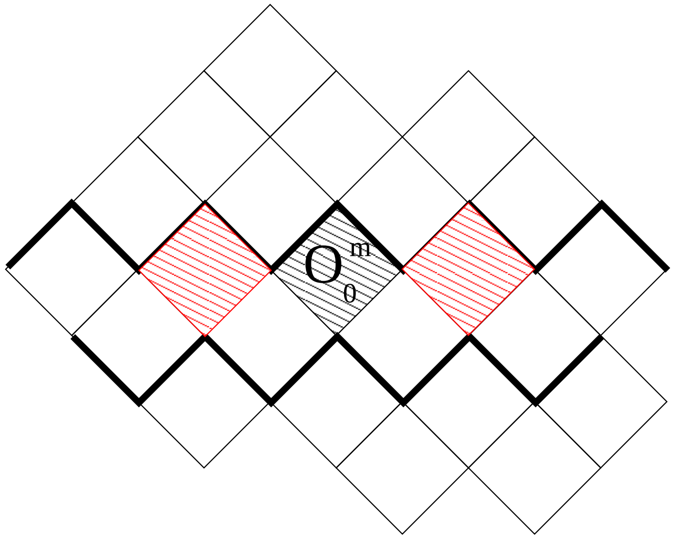
'One-point' algebras



- **Linear basis:** $1, U_0$
- **Minimal projections:** $P = \frac{1}{2}(1 \pm U_0)$
- **Commutation relations:**

$$U_i U_j = \begin{cases} -U_j U_i, & \text{if } |i-j| = \frac{1}{2} \\ U_j U_i, & \text{otherwise} \end{cases}$$

‘One-point’ algebras



- **Linear basis:** $1, U_0$
- **Minimal projections:** $P = \frac{1}{2}(1 \pm U_0)$
- **Commutation relations:**

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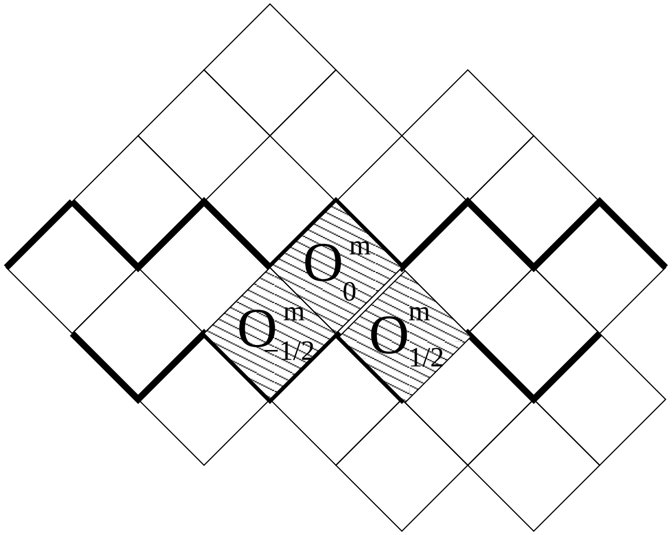
‘Three-point’ algebras



- **Linear basis:**
  $\mathbf{1},\ U_{-\frac{1}{2}},\ U_0,\ U_{\frac{1}{2}},\ iU_{-\frac{1}{2}}U_0,\ iU_0U_{\frac{1}{2}},\ U_{-\frac{1}{2}}U_{\frac{1}{2}},\ U_{-\frac{1}{2}}U_0U_{\frac{1}{2}}$
- **Minimal projections:** $P = P(\vec{\mathbf{n}}), \quad \vec{\mathbf{n}} \in \mathbf{R}^3$
- **Two dimensional projections:** $P = P(\vec{\mathbf{n}}, \vec{\mathbf{n}}')$, $\vec{\mathbf{n}}, \vec{\mathbf{n}}' \in \mathbf{R}^3$