Budapest Semester in Cognitive Science

Introduction to Logic

Day 1. Historical Introduction

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Budapest Semester in Cognitive Science

Introduction to Logic

Day 1. Historical Introduction

Contents:

1. Passages from *Fundamentals of Argumentation Theory* (Eemeren *et al.*, New Jersey: Lawrence Erlabium Associates. 1996.) Pp. 29-37
2. Lewis Carroll: "What the Tortoise said to Achilles" *Mind*, 1895.

Note:

Part of today’s lecture is supported by the material printed in the section "Predicate Logic": Pp. 7-11

Budapest Semester in Cognitive Science

Introduction to Logic

Day 2. Propositional Logic

The reader is largely based on a web course material available at
http://www.cs.odu.edu/~toida/nerzic/content/web_course.html

Contents

Propositions and Truth Values

Propositional logic is a logic at the sentential level. The smallest unit we deal with in propositional logic is a sentence. We do not go inside individual sentences and analyze or discuss their meanings. We are going to be interested only in true or false of sentences, and major concern is whether or not the truth or falsehood of a certain sentence follows from those of a set of sentences, and if so, how. Thus sentences considered in this logic are not arbitrary sentences but are the ones that are true or false. This kind of sentences are called **propositions**.

Propositions are either true or false, but not both. This is the princple of bivalence: any proposition must take exactly one (and neither more nor less) of the two possible truth values. If a proposition is true, then we say it has a **truth value** of "**true**"; if a proposition is false, its truth value is "**false**".

For example, "Grass is green", and "2 + 5 = 5" are propositions. The first proposition has the truth value of "true" and the second "false". But "Close the door", and "Is it hot outside ?"are not propositions.

There are non-classical logical systems where the principle of bivalence does not hold, because some of the propositions does not take any of the truth values 'true' or 'false', or because other truth values are considered. However, we will make the simplest and most intuitive assumption that propositions are either true or false.

Propositions and Connectives

The simplest sentences which are true or false are basic – in other words, atomic – propositions. Larger and more complex sentences are constructed from basic propositions by combining them with **connectives**. Thus **propositions** and **connectives** are the basic elements of propositional logic. Though there are many connectives, we are going to use the following **five basic connectives** here:

NOT, AND, OR, IF_THEN (or IMPLY), IF_AND_ONLY_IF.

They are also denoted by the symbols:

$\sim$, & , v , $\supset$, $\equiv$,

respectively.

Often we want to discuss properties/relations common to all propositions. In such a case rather than stating them for each individual proposition we use variables representing an arbitrary proposition and state properties/relations in terms of those variables. Those variables are called a **propositional variable**. **Propositional variables are also considered a proposition and called a proposition** since they represent a proposition hence they behave the same way as propositions. A proposition in general contains a number of variables. For example (P v Q) contains variables P and Q each of which represents an arbitrary proposition. Thus a proposition takes different values depending on the values of the constituent variables. This relationship of the value of a proposition and those of its constituent variables can be represented by a table. It tabulates the value of a proposition for all possible values of its variables and it is called a **truth table**.

Let us **define the meaning of the five connectives** by showing the relationship between the truth value (i.e. true or false) of composite propositions and those of their component propositions. They are going to be shown using truth table. In the tables P and Q represent arbitrary propositions, and true and false are represented by T and F, respectively.

NOT	
P	**~P**
T	F
F	T

This table shows that if P is true, then (~P) is false, and that if P is false, then (~P) is true.

AND		
P	**Q**	**(P &Q)**
F	F	F
F	T	F
T	F	F
T	T	T

This table shows that (P & Q) is true if both P and Q are true, and that it is false in any other case.

Similarly for the rest of the tables.

OR		
P	**Q**	**(P v Q)**
F	F	F
F	T	T
T	F	T
T	T	T

IMPLIES		
P	**Q**	**(P ⊃ Q)**
F	F	T
F	T	T
T	F	F
T	T	T

IF AND ONLY IF		
P	**Q**	**(P ≡ Q)**
F	F	T
F	T	F
T	F	F
T	T	T

Notes on IMPLY

Note 1

(P ⊃ Q) is True whenever P is False as well as when both P and Q are True. This might be counterintuitive for some people and might be a little difficult to be convinced of. What we are concerned about here is True or False of the statement (P ⊃ Q). You might also look at it this way. We are interested in whether or not the person who made this satement is lying. If the statement is False, then that person is lying. For example consider this sentence:

You get ten thousand dollars from me if I win one million dollars in a lottery.
Here P is "I win one million dollars in a lottery" and Q is "You get ten thousand dollars from me".

If I don't win the lottery (P is False), I don't have to give you ten thousand dollars (Q is False). My statement (P ⊃ Q) is still true if you don't get the money from me when I don't win. I haven't lied to you. This is what "(P ⊃ Q) is True when P is False" means. Similarly for when P and Q are False. On the other hand, if I did win the the lottery and did not give you $10,000, then I have lied to you, that is the statement "You get ten thousand dollars from me if I win one million dollars in a lottery" is not true. That is what "(P ⊃ Q) is False if P is True and Q is False" means.

Note 2

In "If P then Q", P and Q are arbitrary propositions. We are interested in only true or false of (P ⊃ Q) vis-a-vis true or false of P and Q. Thus P and Q may be completely unrelated sentences such as in " If 3 > 1, then ODU is in Norfolk, VA." This proposition is true since both "3 > 1" and "ODU is in Norfolk, VA" are true. As an English sentence this if-then statement is meaningless. However, as a proposition it is legitimate.

Note 3

If-then statements appear in various forms in practice. The following list presents some of the variations. **These are all *logically* equivalent,** that is as far as true or false of statement is concerned there is no difference between them. Thus if one is true then all the others are also true, and if one is false all the others are false.

- **If p , then q.**
- **p implies q.**

- **If p,   q.**
- **p only if q.**
- **p is sufficient for q.**
- **q if p.**
- **q whenever p.**
- **q is necessary for p.**
- **It is necessary for p that q.**

For instance, instead of saying "If she smiles then she is happy", we can say "If she smiles, she is happy", "She is happy whenever she smiles", "She smiles only if she is happy" etc. without changing their truth values.

"Only if" can be translated as "then". For example, "She smiles only if she is happy" is equivalent to "If she smiles, then she is happy". Note that "She smiles only if she is happy" means "If she is not happy, she does not smile", which is the contrapositive of "If she smiles, she is happy". You can also look at it this way: "She smiles only if she is happy" means "She smiles only when she is happy". So any time you see her smile you know she is happy. Hence "If she smiles, then she is happy". Thus they are logically equivalent.

Also **"If she smiles, she is happy" is equivalent to "It is necessary for her to smile that she is happy".** For "If she smiles, she is happy" means "If she smiles, she is *always* happy". That is, she never fails to be happy when she smiles. "Being happy" is inevitable consequence/necessity of "smile". Thus if "being happy" is missing, then "smile" can not be there either. "Being happy" is necessary "for her to smile" or equivalently "It is necessary for her to smile that she is happy".

Syntax of propositions

First it is informally shown how complex propositions are constructed from simple ones. Then more general way of constructing propositions is given.

In everyday life we often combine propositions to form more complex propositions without paying much attention to them. For example combining "Grass is green", and "The sun is red" we say something like "Grass is green and the sun is red", "If the sun is red, grass is green", "The sun is red and the grass is not green" etc. Here "Grass is green", and "The sun is red" are propositions, and form them using connectives "and", "if... then ..." and "not" a little more complex propositions are formed. These new propositions can in turn be combined with other propositions to construct more complex propositions. They then can be combined to form even more complex propositions. This process of obtaining more and more complex propositions can be described more generally as follows:

Let X and Y represent arbitrary propositions. Then
(~X),   (X & Y), (X v Y),   (X ⊃ Y),   and   (X ≡ Y)
are *propositions*.

Note that X and Y here represent an arbitrary proposition.

Example : (P ⊃ (Q v R)) is a proposition and it is obtained by first constructing (Q v R) by applying (X v Y) to propositions Q and R considering them as X and Y, respectively, then by applying (X ⊃ Y) to the two propositions P and (Q v R) considering them as X and Y, respectively.

Note 1: Rigorously speaking X and Y above are place holders for propositions, and so they are not exactly a proposition. They are called a *propositional variable*, and propositions formed from them using connectives are called a *propositional form*. However, we are not going to distinguish them here, and both specific propositions such as "2 is greater than 1" and propositional forms such as (P v Q) are going to be called a proposition.

To convert English statements into a symbolic form, we **restate the given statements using the building block sentences, those for which symbols are given, and the connectives of propositional logic** (**not, and, or, if_then, if_and_only_if**), and **then substitute the symbols for the building blocks and the connectives**.
For example, let **P** be the proposition "It is snowing", **Q** be the proposition "I will go the beach", and **R** be the proposition "I have time". Then first "I will go to the beach if it is not snowing" is restated as "If it is not snowing, I will go to the beach". Then symbols **P** and **Q** are substituted for the respective sentences to obtain **((~P) ⊃ Q)**. Similarly, "It is not snowing and I have time only if I will go to the beach" is restated as "If it is not snowing and I have time, then I will go to the beach", and it is translated as **(((~P) & R) ⊃ Q)**.

Note 2: Although the above syntactical rules are srtict with regards to using parentheses, we will be more generous in using them where it is intuitively possible. We will omit the outmost parentheses: instead of "(~P)" we will write "~P", instead of "(P & Q)" we write "P & Q" We use the parentheses only when the connected proposition is not atomic: "P & Q" negated is "~(P & Q)", etc.

Types of Proposition

Some propositions are always true regardless of the truth value of its component propositions.
For example P v ~P is always true regardless of the value of the proposition P.
A proposition that is always true called a **tautology**.

There are also propositions that are always false such as P & ~P.
Such a proposition is called a **contradiction**.

A proposition that is neither a tautology nor a contradiction is called a **contingency**.
For example P v Q is a contingency. It can be true as well as false.

Whether a contingency is true or false depends on the fact or state of affairs it is meant to describe. In other words its truth value depends on the world. On the other hand, the truth value of a contradiction, or of a tautology, does not depend on the world external to the language, but on the syntactical properties of the language. In a strict sense these propositions do not tell us anything: they have no information content.

With the help pf truth tables we can see whether a proposition is a tautology or a contradiction or a contingency. For example, **P ⊃ (Q ⊃ P)** is shown to be a tautology:

P	Q	Q ⊃ P	P ⊃ (Q ⊃ P)
F	F	T	T
F	T	F	T
T	F	T	T
T	T	T	T

In order to see the truth values of P ⊃ (Q ⊃ P) , we first have to see the truth value of its components (P – first column, Q ⊃ P – third column). By definition of the meaning of 'imply', P ⊃ (Q ⊃ P) is False if and only if P is True and Q ⊃ P is false. Since there is no such case it is True everywhere.

A tautology gives the value True in all the rows of its truth table. A contradiction gives the value False in all the rows of its truth table. Consequently, the negation of a tautology is a contradiction, and vice versa. A contingency has both True and False outputs in its truth table.

Identities

From the definitions (meaning) of connectives, a number of relations between propositions which are useful in reasoning can be derived. Below some of the often encountered pairs of logically equivalent propositions, also called *identities*, are listed.
These identities are used in logical reasoning. In fact we use them in our daily life, often more than one at a time, without realizing it.
If two propositions are logically equivalent, one can be substituted for the other in any proposition in which they occur without changing the logical value of the proposition.

Below "⇔" means that the propositions on the two sides are always true together or false together, i.e. they have the same truth value in all circumstances. Such propositions are identical in the logical sense.

1. P ⇔ P v P – idempotence of v
2. P ⇔ P & P – idempotence of &
3. P v Q ⇔ Q v P – commutativity of v
4. P & Q ⇔ Q & P – commutativity of &
5. P v (Q v R) ⇔ (P v Q) v R – associativity of v
6. P & (Q & R) ⇔ (P & Q) & R – associativity of &
7. ~(P v Q) ⇔ ~P & ~Q – DeMorgan’s Law
8. ~(P & Q) ⇔ ~P v ~Q – DeMorgan’s Law
9. P & (Q v R) ⇔ (P & Q) v (P & R) – distributivity of & over v
10. P v (Q & R) ⇔ (P v Q) & (P v R) – distributivity of v over &
11. P ⇔ ~(~P) – double negation
12. P ⊃ Q ⇔ ~P v Q – implication
13. P ≡ Q ⇔ (P ⊃ Q) & (Q ⊃ P) – biconditional
14. (P & Q) ⊃ R ⇔ P ⊃ (Q ⊃ R) – exportation
15. ((P ⊃ Q) & (P ⊃ ~Q)) ⇔ ~P – absurdity
16. (P ⊃ Q) ⇔ (~Q ⊃ ~P) – contrapositive

All the equvalences can be proven to hold using truth tables as follows. In general two propositions are logically equivalent if they take the same value for each set of values of their variables. Thus to see whether or not two propositions are equivalent, we construct truth tables for them and compare to see whether or not they take the same value for each set of values of their variables.

For example consider the **commutativity of v:** P v Q $\Leftrightarrow$ Q v P

To prove that this equivalence holds, let us construct a truth table for each of the proposition P v Q and Q v P

P	Q	P v Q
F	F	F
F	T	T
T	F	T
T	T	T

P	Q	Q v P
F	F	F
F	T	T
T	F	T
T	T	T

As we can see from these tables P v Q and Q v P take the same value for the same set of values of P and Q. Thus they are (logically) equivalent.

Let us see some example statements in English that illustrate these equvalences.

Examples:

1. What this says is, for example, that "Tom is happy." is equivalent to "Tom is happy or Tom is happy". This and the next identity are rarely used, if ever, in everyday life. However, these are useful when manipulating propositions in reasoning in symbolic form.
2. Similar to 1. above.
3. What this says is, for example, that "Tom is rich or (Tom is) famous." is equivalent to "Tom is famous or (Tom is) rich".
4. What this says is, for example, that "Tom is rich and (Tom is) famous." is equivalent to "Tom is famous and (Tom is) rich".
5. What this says is, for example, that "Tom is rich or (Tom is) famous, or he is also happy." is equivalent to "Tom is rich, or he is also famous or (he is) happy".
6. Similar to 5. above.
7. For example, "It is not the case that Tom is rich or famous." is true if and only if "Tom is not rich and he is not famous."
8. For example, "It is not the case that Tom is rich and famous." is true if and only if "Tom is not rich or he is not famous."
9. What this says is, for example, that "Tom is rich, and he is famous or (he is) happy." is equivalent to "Tom is rich and (he is) famous, or Tom is rich and (he is) happy".

10. Similarly to 9. above, what this says is, for example, that "Tom is rich, or he is famous and (he is) happy." is equivalent to "Tom is rich or (he is) famous, and Tom is rich or (he is) happy".
11. What this says is, for example, that "It is not the case that Tom is not 6 foot tall." is equivalent to "Tom is 6 foot tall."
12. For example, the statement "If I win the lottery, I will give you a million dollars." is not true, that is, I am lying, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Similarly, the statement "I don't win the lottery or I give you a million dollars." is false, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Thus these two statements are logically equivalent.
13. What this says is, for example, that "Tom is happy if and only if he is healthy." is logically equivalent to ""if Tom is happy then he is healthy, and if Tom is healthy he is happy."
14. For example, "If Tom is healthy, then if he is rich, then he is happy." is logically equivalent to "If Tom is healthy and rich, then he is happy."
15. For example, if "If Tom is guilty then he must have been in that room." and "If Tom is guilty then he could not have been in that room." are both true, then there must be something wrong about the assumption that Tom is guilty.
16. For example, "If Tom is healthy, then he is happy." is logically equivalent to "If Tom is not happy, he is not healthy."

Here an example is presented to show how the equvalences can be used to prove some useful results.

$\sim(P \supset Q) \Leftrightarrow P \;\&\; \sim Q$

What this means is that the **negation of "if P then Q"** is **"P** but **not Q"**. For example, if you said to someone "If I win a lottery, I will give you $100,000." and later that person says "You lied to me." Then what that person means is that you won the lottery but you did not give that person $100,000 you promised.

To prove this, first let us get rid of $\supset$ using one of the identities**:** $P \supset Q \Leftrightarrow \sim P \vee Q$
That is, $\sim(P \supset Q) \Leftrightarrow \sim(\sim P \vee Q)$
Then by De Morgan, it is equivalent to $\sim(\sim P) \;\&\; \sim Q$, which is equivalent to $P \;\&\; \sim Q$, since the double negation of a proposition is equivalent to the original proposition as seen in the identities.

Implications

The following implications are some of the relationships between propositions that can be derived from the definitions(meaning) of connectives. "$\Rightarrow$" below means that *if* the proposition on the left is true, *then* the one on the right must also be true.

These implications are used in logical reasoning. When the right hand side of these implications is substituted for the left hand side appearing in a proposition, the resulting proposition is implied by the original proposition, that is, one can deduce the new proposition from the original one.

First some implications are listed, then examples to illustrate them are given.

List of Implications:

1. P $\Rightarrow$ P v Q – addition
2. P & Q $\Rightarrow$ P – simplification
3. P & (P $\supset$ Q) $\Rightarrow$ Q – modus ponens
4. ~Q & (P $\supset$ Q) $\Rightarrow$ ~P – modus tollens
5. ~P & (P v Q) $\Rightarrow$ Q – disjunctive syllogism
6. (P $\supset$ Q) & (Q $\supset$ R) $\Rightarrow$ P $\supset$ R – hypothetical syllogism

Examples:

1. For example, if the sun is shining, then certainly the sun is shining or it is snowing. Thus "if the sun is shining, then the sun is shining or it is snowing." "If $0 < 1$, then $0 \leq 1$ or a similar statement is also often seen.
2. For example, if it is freezing and (it is) snowing, then certainly it is freezing. Thus "If it is freezing and (it is) snowing, then it is freezing."
3. For example, if the statement "If it snows, the schools are closed" is true and it actually snows, then the schools are closed.
   This implication is the basis of all reasoning. Theoretically, this is all that is necessary for reasoning. But reasoning using only this becomes very tedious.
4. For example, if the statement "If it snows, the schools are closed" is true and the schools are not closed, then one can conclude that it is not snowing.
   Note that this can also be looked at as the application of the contrapositive and modus ponens. That is, P $\supset$ Q is equivalent to ~Q $\supset$ ~P. Thus if in addition ~Q holds, then by the modus ponens, ~P is concluded.
5. For example, if the statement "It snows or (it) rains." is true and it does not snow, then one can conclude that it rains.
6. For example, if the statements "If the streets are slippery, the school buses can not be operated." and "If the school buses can not be operated, the schools are closed." are true, then the statement "If the streets are slippery, the schools are closed." is also true.

All implications can be shown valid or invalid by using truth tables. Examples:

P & (P $\supset$ Q) $\Rightarrow$ Q

P	**Q**	**P $\supset$ Q**	**P & (P $\supset$ Q)**
F	F	T	F
F	T	T	F
T	F	F	F
T	T	T	T

The implication says that *if* P & (P $\supset$ Q) is true, *then* Q is also true. The truth table shows that P & (P $\supset$ Q) is true in only one case, but in that case Q is also true. So the implication us valid.

Let's see a bit more difficult example:

$(P \supset Q) \& (Q \supset R) \Rightarrow P \supset R$

P	**Q**	**R**	**P ⊃ Q**	**Q ⊃ R**	**(P ⊃ Q) & (Q ⊃ R)**	**P ⊃ R**
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	F	F
T	F	T	F	T	F	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Here the truth table has 8 rows because the truth values of P, Q and R can be combined in 8 different ways. The values for $P \supset Q$, $Q \supset R$ and $P \supset R$ can be writen from the values of P, Q and R by using the definition of 'implies', while the truth values of $(P \supset Q) \& (Q \supset R)$ can be written from the values of $P \supset Q$ and $Q \supset R$ by using the definition of 'and'. We see that in any case when $(P \supset Q) \& (Q \supset R)$ is true (rows 1, 2, 4, 8), $P \supset R$ is also true. In other words, there is no such case when $(P \supset Q) \& (Q \supset R)$ is true and $P \supset R$ is false. Note that the definition does not care about the truth value of $P \supset R$ when $(P \supset Q) \& (Q \supset R)$ is false: then it can both be true (rows 3, 6) and be false (rows 5, 7). The only thing we want to know is that when the left hand proposition is true, then the right hand one is also true, in this order.

Inferences

An inference is a relation between a set of propositions (premises) and a proposition (conclusion). An inference is **valid** when the following relation between the truth values holds: ***if*** **all the premises are true,** ***then*** **the conclusion is also true**. In other words, it is impossible for the premises to be true and for the coclusion to be false at the same time. This is how we ensure that the truth of the conclusion follows from the truth of the premises.

A valid inference is written in the following way:

P, Q, R ⇒ C ,

where P, Q and R are the premises and C is the conclusion.

Note that the sign '⇒' here is the same as in implications. An implication is an inference with one premise (see their definitions). Of course, there can be more than one premise in an inference: as many as you want.

Technically speaking, the minimal number of premises you need for a valid inference is zero. Any tautology can be seen as the conclusion of an inference from zero premises. It is because in case of a tautology, for any set of premises it is impossible for them to be true and the tautology to be false simultanously, since the teutology itself can never be false. Thus the set of premises can be anything, even an empty set. For this reason, tautologies are often indicated in this way:

$\Rightarrow$ **P ⊃ P** ,

showing that such a proposition follows from any premises whatsoever.

The validity of inferences in propositional logic can be checked by using truth tables. For example:

1. P ⊃ Q, P ⇒ Q

P	Q	P ⊃ Q	
T	T	T	✓
T	F	F	
F	T	T	
F	F	T	

What we have to see if the following condition of valid inference holds: *if* the premises are true, *then* the conclusion must also be true. The premises are **P** and **P ⊃ Q** (see first and third columns). There is only one case when both are true (see first row), but then the conclusion (second column) is also true. Therefore the inference is valid.

For the same reason, the following is invalid:

2. P ⊃ Q, Q ⇒ P

P	Q	P ⊃ Q	
t	t	T	✓
t	f	F	
f	t	T	∅
f	f	T	

The premises (second and third columns) are both true in two cases (first and third rows). Out of these, the conclusion (first column) is true in the first row BUT false in the third. So it does not hold that *if* the premises are true, *then* the conclusion must also be true.

Reasoning with Propositions

Logical reasoning is the process of drawing conclusions from premises using rules of inference. The basic inference rule is **modus ponens**. It states that if both P ⊃ Q and P hold, then Q can be concluded, and it is written as

P
<u>P ⊃ Q</u>
Q

Here the propositions above the line are **premises** and the line below it is the **conclusion** drawn from the premises. For example if "if it rains, then the game is not played" and "it rains" are both true, then we can conclude that the game is not played.

Example of Inferencing

Consider the following argument**:**

1. Today is Tuesday or Wednesday.

2. But it can't be Wednesday, since the doctor's office is open today, and that office is always closed on Wednesdays.

3. Therefore today must be Tuesday.

This sequence of reasoning (inferencing) can be represented as a series of application of modus ponens to the corresponding propositions as follows.

The modus ponens is an inference rule which deduces **Q** from **P ⊃ Q** and **P**.

T: Today is Tuesday.

W: Today is Wednesday.

D: The doctor's office is open today.

C: The doctor's office is always closed on Wednesdays.

The above reasoning can be represented by propositions as follows.

1. T v **W**

2. D
C

~W
----
3. T

To see if this conclusion **T** is correct, let us first find the relationship among **C**, **D**, and **W**:

C can be expressed using **D** and **W**. That is, restate **C** first as the doctor's office is always closed if it is Wednesday. Then **C ≡ (W ⊃ ~D)** Thus substituting **(W ⊃ ~D)** for **C**, we can proceed as follows.

D
W ⊃ **~D**

~W

which is correct by modus tollens. From this **~W** combined with **T** v **W** of **1.** above,

~W
T v **W**

T

which is correct by disjunctive syllogism. Thus we can conclude that the given argument is correct.
To save space we also write this process as follows eliminating one of the **~W**'s:

D
W ⊃ **~D**

~W
T v **W**

T

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The reader is partly based on a web course material available at
http://www.cs.odu.edu/~toida/nerzic/content/web_course.html

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Introduction to Predicate Logic

The propositional logic is not powerful enough to represent all types of assertions that are used in everyday language, or to express certain types of relationship between propositions such as equivalence. The pattern involved in the following logical equivalences can not be captured by the propositional logic:

"Not all birds fly" is equivalent to "Some birds don't fly".
"Not all integers are even" is equivalent to "Some integers are not even".
"Not all cars are expensive" is equivalent to "Some cars are not expensive",

Each of those propositions is treated independently of the others in propositional logic. For example, if P represents "Not all birds fly" and Q represents "Some integers are not even", then there is no mechanism in propositional logic to find out tha P is equivalent to Q. Hence to be used in inferencing, each of these equivalences must be listed individually rather than dealing with a general formula that covers all these equivalences collectively and instantiating it as they become necessary, if only propositional logic is used.

Thus we need more powerful logic to deal with these and other problems. The predicate logic is one of such logic and it addresses these issues among others.

To cope with deficiencies of propositional logic we introduce two new features: predicates and quantifiers.

Predicates

A **predicate** is a verb phrase template that describes a property of objects, or a relationship among objects represented by the variables.

For example, the sentences "The car Tom is driving is blue", "The sky is blue", and "The cover of this book is blue" come from the template "is blue" by placing an appropriate noun/noun phrase in front of it. The phrase **"is blue"** is a predicate and it describes the property of being blue. Predicates are often given a **name**. For example any of "is_blue", "Blue" or "B" can be used to represent the predicate "is blue" among others. If we adopt B as the name for the predicate "is_blue", sentences that assert an object is blue can be represented as "B(x)", where x represents an arbitrary object. B(x) reads as "x is blue".

Similarly the sentences "John gives the book to Mary", "Jim gives a loaf of bread to Tom", and "Jane gives a lecture to Mary" are obtained by substituting an appropriate object for variables *x*, *y*, and *z* in the sentence "*x* gives *y* to *z*". The template "... gives ... to ..." is a predicate and it describes a relationship among three objects. This predicate can be represented by Give(x, y, z) or G(x, y, z), for example.

Note: The sentence "John gives the book to Mary" can also be represented by another predicate such as "gives a book to". Thus if we use B(x, y) to denote this predicate, "John gives the book to Mary" becomes B(John, Mary). In that case, the other sentences, "Jim gives a loaf of bread to Tom", and "Jane gives a lecture to Mary", must be expressed with other predicates.

In the followings, predicates will be denoted by upper case letters (A, B, C,…). After these letters we put the so-called argument(s) of the predicate in parentheses, denoted by lower case letters (a, b, c, …). Thus the sentence "Many is happy" will be written as H(m).

Sentences such as above are, like the propositions of propositional logic, either true or false. However, we will not call them propositions because, in the symbolic language of predicate logic, we express their internal logical structure, contrary to what we did in propositional logic. The sentences formalized in the language of predicate logic will be called **formulas**.

H(x) expressing that "x is happy" does not have a truth value, since x represents an arbitrary object. Thus we have two types of signs which can be written as arguments of predicates:

- the so-called names or terms: a, b, c… – always represent a concrete person or object
- the so-called variables: x, y, z,… – always represents an arbitrary object.

A predicate filled with the necessary number of names gives a formula. It has a truth value of its own, therefore it is called a closed formula. On the other hand, a predicate filled with insufficient number of names and with variables for the lack of names is an open formula. Open formulas (such as F(x), G(a, x), H(x, y) etc.) have no truth values. For example, "2+4=6" is a tue sentence, while "2+4=x" is neither true nor false.

Note: Names do not have to be proper names such as John. Any expression successfully identifying a single person or object is a name in the logical sense. For example, instead of 'Julius Caesar' you can say 'Brutus' father', instead of 'London' you can say 'the capital of Britain', instead of '6' you can say '2+4', etc.

Formulas with Predicates

When you have a predicate filled with the appropriate number of names or variables, you have a formula. Of course, such formulas can be taken as abbriviated by atomic propositions in propositional logic. The connectives of propositional logic can be used to connect formulas of predicate logic as well. Thus "If John is rich than Mary is happy" can be written as:

R(j) ⊃ H(m)

where R and H are predicates (rich, happy), and j and m are names (John, Mary).

The rules for building formulas are:

- If **A** is a 1-place predicate and **a** is a name, **x** is a variable, then **A(a)** is a formula and **A(x)** is a formula. Similarly, if **B** is a 2-place predicate and **a, b** are names, **x, y** are variables, then **B(a, b), B (x, y), B(a, x)** etc. are formulas. Etc.
- If **X, Y** are formulas, then **~X, X ⊃ Y, X v Y, X & Y, X ≡ Y** are also formulas.

Quantification

An open formula has no truth value. In everyday language we don't really use open formulas. But in predicate logic, they enable us to express true or false sentences about, not concrete objects, but classes of objects. For this, we need as new tools the so-called **quantifiers**.

The Universal Quantifier

The expression: $\forall$x P(x), denotes the **universal quantification** of the atomic formula P(x). Translated into the English language, the expression is understood as: "*For all x, P(x)* holds", "*for each x, P(x)* holds" or "*for every x, P(x)* holds". $\forall$ is called the **universal quantifier**, and $\forall$*x* means all the objects x in the universe. If this is followed by P(x) then the meaning is that P(x) is true for every object *x* in the universe. For example, "All cars have wheels" could be transformed into the propositional form, $\forall$x P(x), where:

- P(x) is the predicate denoting: ***x* has wheels**, and
- the universe of discourse is only populated by cars.

Note: The universe of discourse, also called **universe**, is the set of objects of interest. The propositions in the predicate logic are statements on objects of a universe. The universe is thus the domain of the (individual) variables. It can be the set of real numbers, the set of integers, the set of all cars on a parking lot, the set of all students in a classroom etc. The universe is often left implicit in practice. But it should be obvious from the context.

Universal Quantifier and **Connective AND**
If all the elements in the universe of discourse can be listed then the universal quantification **$\forall$x P(x)** is equivalent to the conjunction: **$P(x_1)$ & $P(x_2)$ & $P(x_3)$ &... &$P(x_n)$** .
For example, in the above example of $\forall$x P(x), if we knew that there were **only** 4 cars in our universe of discourse (c1, c2, c3 and c4) then we could also translate the statement as:
P(c1) & P(c2) & P(c3) & P(c4).

The Existential Quantifier

The expression $\exists$xP(x) denotes the **existential quantification** of P(x). Translated into the English language, the expression could also be understood as: "There exists an *x* such that P(x)" or "There is at least one *x* such that P(x)". $\exists$ is called the **existential quantifier**, and $\exists$x means at least one object *x* in the universe. If this is followed by P(x) then the meaning is that P(x) is true for at least one object *x* of the universe. For example, "*Someone loves you*" could be transformed into the propositional form $\exists$xP(x), where:

- P(x) is the predicate meaning: ***x* loves you**,
- The universe of discourse contains (but is not limited to) all living creatures.

Existential Quantifier and **Connective OR**
If all the elements in the universe of discourse can be listed, then the existential quantification $\exists$xP(x) is equivalent to the disjunction: **$P(x_1)$ v $P(x_2)$ v $P(x_3)$ v… v $P(x_n)$.**
For example, in the above example of $\exists$xP(x), if we knew that there were **only** 5 living

creatures in our universe of discourse (say: me, he, she, rex and fluff), then we could also write the statement as: P(me) v P(he) v P(she) v P(rex) v P(fluff).

How to read quantified formulas

When reading quantified formulas in English, **read them from left to right.** $\forall x$ can be read as "for every object ***x*** in the universe the following holds" and $\exists$***x*** can be read as "there erxists an object ***x*** in the universe which satisfies the following" or "for some object ***x*** in the universe the following holds". Those do not necessarily give us good English expressions. But they are where we can start. Get the correct reading first then polish your English without changing the truth values.

For example, let the universe be the set of airplanes and let F(x, y) denote "x flies faster than y". Then

- $\forall x\forall yF(x, y)$ can be translated initially as "For every airplane ***x*** the following holds: ***x*** is faster than every (any) airplane ***y***". In simpler English it means "Every airplane is faster than every airplane (including itself !)".
- $\forall x\exists yF(x, y)$ can be read initially as "For every airplane ***x*** the following holds: for some airplane ***y***, ***x*** is faster than ***y***". In simpler English it means "Every airplane is faster than some airplane".
- $\exists x\forall yF(x, y)$ represents "There exist an airplane ***x*** which satisfies the following: (or such that) for every airplane ***y***, ***x*** is faster than ***y***". In simpler English it says "There is an airplane which is faster than every airplane" or "Some airplane is faster than every airplane".
- $\exists x\exists yF(x, y)$ reads "For some airplane ***x*** there exists an airplane ***y*** such that ***x*** is faster than ***y***", which means "Some airplane is faster than some airplane".

When more than one variables are quantified in a formula such as $\forall x\exists yF(x, y)$, they are applied from the inside, that is, the one closest to the atomic formula is applied first. Thus $\forall x\exists yF(x, y)$ reads $\forall x\ (\ \exists yF(x, y)\ )$, and we say " for every ***x*** there exists an ***y*** such that F(x, y) holds".

The positions of the same type of quantifiers can be switched without affecting the truth value as long as there are no quantifiers of the other type between the ones to be interchanged. Thus
$\forall x\forall yF(x, y) \Leftrightarrow \forall y\forall xF(x, y)$ and
$\exists x\exists yF(x, y) \Leftrightarrow \exists y\exists xF(x, y)$

However, the positions of different types of quantifiers can **not** be switched. For example $\forall x\exists yF(x, y)$ is *not* equivalent to $\exists y\forall xF(x, y)$. For let F(x, y) represent $x < y$ for the set of numbers as the universe, for example. Then $\forall x\exists yF(x, y)$ reads "for every number ***x***, there is a number ***y*** that is greater than ***x***", which is true, while $\exists y\forall xF(x, y)$ reads "there is a number that is greater than every (any) number", which is not true.

Formulas with Quantifiers

Now we have one more rule for building formulas, which must be added to the previous ones:

- If X is a formula and x is a variable, then $\forall xX$ and $\exists xX$ are also formulas.

Technically speaking, it does not matter whether or not the quantified formula is closed, i.e. if it contains variables. But for the purposes of translating everyday language sentences to predicate logic formulas, we meet only such cases where the variable following the quantifier occurs in the quantified formula. For example, it is meaningful to say that "For every *x*, *x* is greater than zero", while it is without meaning to say that "For every *x*, one is greater than zero".

Categorical Formulas

The most ancient logical theory, developed by Aristotle in the 4th century BC, was about the so-called categorical sentences. The four types were:

- Universal assertive: 'All philosophers are wise'
- Particular assertive: 'Some philosophers are wise'
- Universal negative: 'No philosophers are wise'
- Particular negative: 'Some philosophers are not wise'

How do we represent these sentences in predicate logic? One solution is to write them as:

$\forall xW(x)$, $\exists xW(x)$, $\forall x{\sim}W(x)$, $\exists x{\sim}W(x)$,

and add that the universe is the set of all philosophers. However, sometimes it is impossible to adjust the universe to the domain of quantification, like in this case: 'All philosophers are wise and all politicians are liars' – here the universe should change between the two parts of the compound assertion $\forall xW(x) \& \exists xL(x)$ and it is not allowed by the rules of logic. Instead, we have to express with the syntactical structure of the formula that 'wise' refers to philosophers and 'liars' refers to politicians.

In predicate logic, 'wise' is represented with a one-place predicate and 'philosopher' with another one-place predicate: W(x) , P(x). Since predicates filled with their arguments are formulas, we guess that there will be a connective between there formulas. Which one? The sentence 'All philosophers are wise' can be expressed in this way: '*if* someone is a philosopher *then* s/he is wise' – $P(x) \supset W(x)$. Using the universal quantifier we can express that this assertion is true to every *x*:

$\boldsymbol{\forall x(P(x) \supset W(x))}$

Now we don't have to speak about the universe, and the sentence 'All philosophers are wise and all politicians are liars' can be translated as $\forall x(P(x) \supset W(x)) \& \forall x(Po(x) \supset L(x))$.

The sentence 'Some philosophers are wise' makes another connection between P(x) and W(x). It says that 'There are *some* who are (both) philosophers *and* wise' , so both the quantifier and the connective are different:

$\exists x(P(x) \ \& \ W(x))$

The sentecne saying that 'No philosophers are wise' translates as '*if* someone is a philsosopher *then* s/he is *not* wise: $\forall x(P(x) \supset \sim W(x))$, while the sentecne 'Some philosophers are not wise' is written as $\exists x(P(x) \ \& \ \sim W(x))$.

To sum up, the basic types of categorical propositions can be formalized in the following forms:

- Universal assertive: **$\forall x(A(x) \supset B(x))$**
- Particular assertive: **$\exists x(A(x) \ \& \ B(x))$**
- Universal negative: **$\forall x(A(x) \supset \sim B(x))$**
- Particular negative: **$\exists x(A(x) \ \& \ \sim B(x))$**

Identities of Quantified Formulas

What about the negation of quantified formulas? For example, to say that "there is no such thing as an orc" is equivalent with the statement that "for every thing in the world, it is not an orc". With formulas:

$\sim\exists x O(x) \ \Leftrightarrow \ \forall x \sim O(x)$.

In other words, the negation of an existentially quantified formula is equivalent to the universal quantification of the negated formula.

On the other hand, the sentence "not everything is material" seems to mean the same as "something is not material":

$\sim\forall x M(x) \ \Leftrightarrow \ \exists x \sim M(x)$.

The negation of a universially quantified formula is equivalent to the existential quantification of the negated formula.

Using the rules of duble negation we can also write:

$\exists x O(x) \ \Leftrightarrow \ \sim\forall x \sim O(x)$ ("There are orcs" – "Not everything is not an orc")
$\forall x M(x) \ \Leftrightarrow \ \sim\exists x \sim M(x)$ ("Everything is material" – "Nothing is not material")

Identities of Categorical Formulas

In case of a universally assertive formula, we can write the above idenity as

$\forall x(A(x) \supset B(x)) \Leftrightarrow \sim\exists x\sim(A(x) \supset B(x))$.

Using one of the identities of propositional logic: $\sim(P \supset Q) \Leftrightarrow P \& \sim Q$,
we can write the right side this identity in this way:

1. **$\forall x(A(x) \supset B(x)) \Leftrightarrow \sim\exists x(A(x) \& \sim B(x))$**

Similarly, by using the above identities and the identities of propsitional logic, we can come to the following identities:

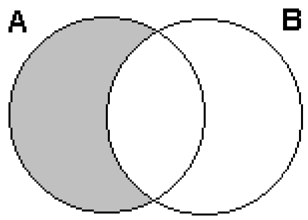
2. **$\exists x(A(x) \& B(x)) \Leftrightarrow \sim\forall x(A(x) \supset \sim B(x))$**
3. **$\sim\forall x(A(x) \supset B(x)) \Leftrightarrow \exists x(A(x) \& \sim B(x))$**
4. **$\sim\exists x(A(x) \& B(x)) \Leftrightarrow \forall x(A(x) \supset \sim B(x))$**

Examples:

1. "Every philosopher is wise" is the same as "These is no philosopher who is not wise".
2. "Some politicians are liars" is the same as "Not every politician is not a liar".
3. "Not all hobbits are coward" is the same as "Some hobbits are not coward".
4. "There is no handsome orc" is the same as "All orcs are not hadsome".

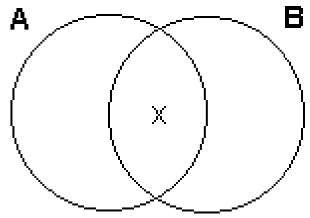
Categorical Formulas and Venn Diagrams

Note that these categorical sentences can be understood to be about relations of classes. The sentence 'All philosophers are wise' says that the class of philosophers is contained by the class of wise people. In other words, everyone who belongs to the class of philosophers belongs to the class of wise people too, and there is no one belonging to the class of philosophers who does not belong to the class of wise people. This can be expressed with this simple figure:



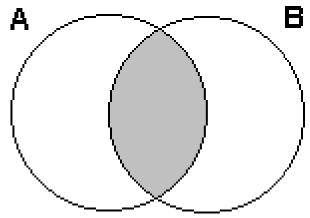
If *A* is the class of philosophers, and *B* is the class of wise people, then the formula **$\forall x(A(x) \supset B(x))$** , which is equvalent to $\sim\exists x(A(x) \& \sim B(x))$, simply states that the part of *A* which does not belong to *B* is empty – denoted by the dark area.

"Some politicians are liars" says that the class of politicians and the class of liars have elements in common. That is:



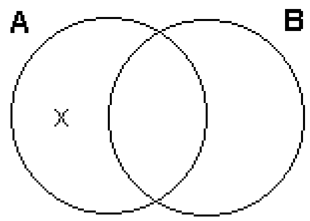
If *A* is the class of philosophers and *B* is the class of liars, then the formula **∃x(A(x) & B(x))** states that the intersection of *A* and *B* is nonempty, since some objects belong to there – denoted by an *x*.

"All orcs are not hadsome" says that the class of orcs and the class of handsome creatures have no element in common. That is:



If *A* is the class of orcs and *B* is the class of handsome creatures, then the formula **∀x(A(x) ⊃ ~B(x))** , which is equivalent to ~∃x(A(x) & B(x)) , states that the intersection of *A* and *B* is empty, and no objects belong to there – denoted by the dark area.

Finally, "Some hobbits are not coward" state that class of hobbits is not contained by the class of coward creatures. In other words, some elements of the class of hobbits do not belong to the class of cowards. That is:



If *A* is the class of hobbits and *B* is the class of cowards, then the formula **∃x(A(x) & ~B(x))** states that the part of *A* which does not belong to *B* is nonempty – denoted by an *x* which belongs to there.

Note: These figures illustrate that some categorical sentences are negations of each other. For example, it is clear that, when *A* and *B* refer to the same predicates, ∀x(A(x) ⊃ B(x)) and ∃x(A(x) & ~B(x)) state the opposite, since the latter is equivalent to ~∀x(A(x) ⊃ B(x)) , which is the negation of the former. On the figures we can see that the former displays an empty area where the latter displays a nonempty one. The same holds for ∀x(A(x) ⊃ ~B(x)) and ∃x(A(x) & B(x)) . Since these pairs of sentences can never be simultaneously true (nor simultaneously false), traditionally we say that they contradict one another. That also means that if P is a formula that is logically equivalent to Q , then both pairs P and ~Q , and ~P and Q , are contradictory formulas.

Checking Categorical Syllogisms

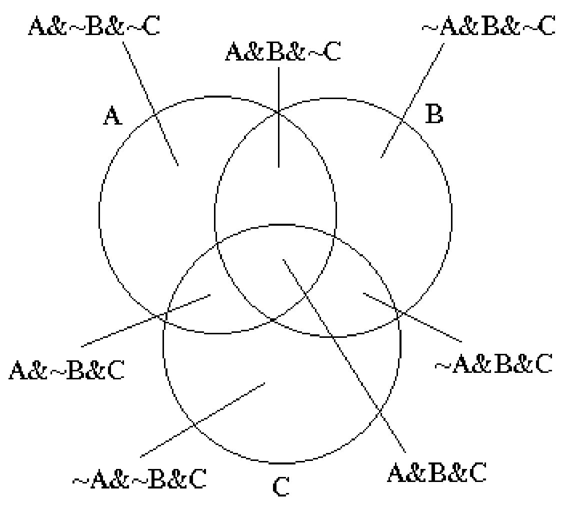
Categorical syllogisms are those inferences, studied by Aristotle, which validly draw a categorical conclusion from two categorical premises having one predicate term in common. For example:

> No orcs are handsome.
> All towerguards are orcs.
> ------------------------------------
> No towerguards are handsome.

How are we to check the validity of such an inference? One way would be to find a method, such as the one using truth tables in propositional logic, which can tell the validity of every inference in predicate logic, including the categorical syllogisms. Such methods exist, but because they are too difficult to study here we will discuss a simpler method which is suitable for only categorical syllogisms. This is based on Venn diagrams.

Step 1

The two premises and the conclusion altogether contain three predicate terms. The three terms are represented by three circular shapes, each having common parts with all the others and also having separate parts. Here all the possibilities can be illustrated:



Step 2

The contents of the premises are indicated in the way discussed earlier. The order of using the premises is important: if one premise states that an area is empty (universal assertive or universal negative), and the other premise states that an area is nonempty (particular assertive or particular negative), then we start with the former one and continue with the latter one. In other cases the order does not matter.

Step 3

The validity or invalidity of inferencing the conclusion is shown by the results of Steps 1 and 2.

Examples

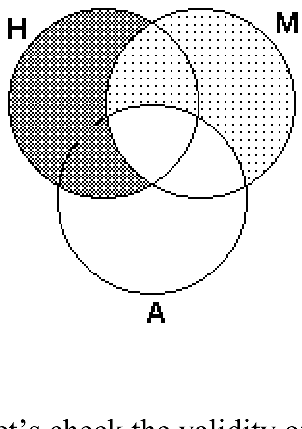
1. Let’s check the above syllogism speaking about orcs, handsome creatures and towerguards!

Explanation:
The content of the first premise is represented by the dark region: O&H is empty, since there is no handsome orc. The content of the second premise is represented by the lighter grey region: T&~O is empty, since there is no towerguard who is not an orc. Now the conclusion says that there is no handsome towerguard, i.e. H&T is empty. This can be read from the figure, since H&T&O was found empty by the first premise, and H&T&~O was found empty by the second premise. The syllogism is valid.

2. Let’s check the validity of the following, very simple and intuitively clear, syllogism:

All hedgehogs are mammals.
All mammals are animals.

All hedgehogs are animals.

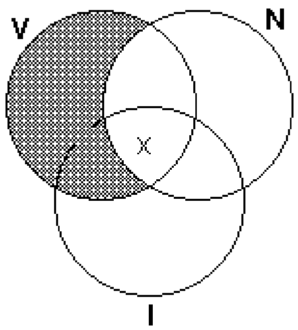


Explanation:
The content of the first premise is represented by the dark region: H&~M is empty, since there is no hedgehog which is not a mammal. The content of the second premise is represented by the lighter grey region: M&~A is empty, since there is no mammal that is not an animal. The validity of the conclusion is affirmed by the figure, since it says that H&~A is empty – indeed: H&~A&~M was found empty by the first premise, H&~A&M was found empty by the second premise.

3. Let’s check the validity of the following syllogism containing a nonempty premise:

Every vampire is a night-creature.
There are immortal vampires.

There are immortal night creatures.



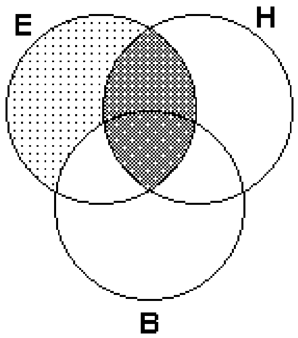
Explanation:
The content of the first premise is represented by the dark region: V&~N is empty, since there is no vampire which is not a night-creature. The content of the second premise is represented by the region with the *x* in it: V&I is nonempty, since there exist immortal vampires. Since V&I&~N was found empty by the first premise, we have to put the *x* in V&I&N. The validity of the conclusion is affirmed by the figure, since it says that I&M is nonempty – and indeed, I&M&V was found nonempty by the second premise.

Note: The example illustrates why we needed the restriction concerning the order of representing the premises. If we started with the second premise, we would only know that somewhere in V&I there is an *x*, but we could not decide whether to put the *x* in V&I&~N or in V&I&N. But since the first can be ruled out by the first premise, it is better to start with it.

4. Let's check the validity of the following, seemingly valid but eventually invalid, syllogism:

No elves are humble.
All elves are brave.

Some are brave but not humble.



Explanation:
The content of the first premise is represented by the dark region: E&H is empty, since there is no humble elf. The content of the second premise is represented by the lighter grey region: E&~B is empty, since there is no elf who is not brave. Now the conclusion says that there is something in B&~H. But on our figure there is no *x*, i.e. no premise have stated the nonemptiness of any area. There *could be*: it's left possible by the premises, but still there *isn't*. So the inference is invalid.

Note: In the above example both the premises are such that they state the emptiness of a certain area. None of them states nonemptiness. Now if that is the case, then they surely do not imply nonemptiness, and no conclusion stating existence can be drawn. Why do some of us feel that the inference is valid? We assume that those things we speak about do exist: If we speak of elves, then elves surely exist. But it is clearly not the case. If there *were* elves, then the only place left for an *x* within E would be E&B&~H, and then the conclusion stating the nonemptiness of B&~H would be justified.

Budapest Semester in Cognitive Science

Introduction to Logic

Day 4. Informal Logic

Contents: