Outcomes in branching space-time and GHZ-Bell theorems

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Abstract

We provide an algebraic framework in which subluminal causation can be analyzed. The framework merges Belnap's 'outcomes in branching time' with his 'branching space-time' (BST). We show that an important structure in BST, called 'the family of outcomes of an event', is a boolean algebra. We next define the non-stochastic common cause and analyze the GHZ-Bell theorems. We prove that there is no common cause that accounts for the results of the GHZ-Bell experiment but construct common causes for two other quantum mechanical setups. Finally, we investigate why some setups allow for common causes whereas other setups do not.

1 Introduction

A significant part of philosophers' curiosity about Bell-type theorems seems to derive from a lesson that these theorems allegedly teach about the causal structure of the world. The theorems prove a significant fact, namely that some experimentally well corroborated quantum predictions cannot be reproduced by so-called local contextual hidden variable models. Technical as the result is, it may nevertheless reveal something about causality, since one premise of the theorems bears a close resemblance to what is known as 'Reichenbach's common cause principle' (Reichenbach [1956]). This principle, or rather a part of it, requires that a common cause CC of two stochastically correlated events A and B screen one off the other, that is, CC should satisfy: $prob(A\&B/CC) = prob(A/CC) \times prob(B/CC)$. Accordingly, if it is accepted that, indeed, the common cause principle is operative in the derivations of the Bell-type theorems and the principle correctly captures the phenomena of common causation, then Bell-type theorems put an end to the causal order, or, at least, rejects an ideal of causal explanation. Since the theorems are experimentally confirmed, it appears that finally a highly metaphysical issue can be experimentally tested.[1]

Recently, however, the issue of how much causality is involved in the derivations of Bell-type theorems has become rather controversial. Thus, Nancy Cartwright argues that screening-off is not a sufficiently general condition on common causes that operate in a chancy way; she shows, moreover, that what she believes to be the general common cause requirement does not ensure the derivation of Bell-type theorems. This requirement can nevertheless be strengthened to Reichenbach's principle if it is assumed that causes propagate contiguously, which is not a very reasonable assumption in quantum phenomena (Cartwright [1989], Ch. 6). Critics of Cartwright argue that her models are nevertheless committed to Reichenbach's principle.[2] Others show that the theorems may be derived from a somewhat different condition, still motivated by some facts about common causation.[3] Clearly, we are far from being unanimous as to what Bell-type theorems say about causation.

One natural way out of this predicament would be to clarify the nature of probabilistic causation and its instances in Bell-type setups. Far from being opposed to this program, we are rather skeptical about its result, since, as it appears to us, the two notions involved: probability and causality, are among the least clear that humans have. Fortunately, among the Bell-type theorems there is a variety, named after its inventors (Greenberger, Horne, and Zeilinger [1989]) GHZ-Bell theorems, that are non-stochastic but, like their stochastic cousins, prove that reconstructing quantum predictions by local contextual hidden variable models is impossible. However, because of the nature of these theorems, since the only probabilities involved are zero and one, the common cause principle is reduced to triviality.

Accordingly, one substantial task of our project is to develop a framework in which subluminal causality can be analyzed in space-time structures. Since Bell-type arguments invariably use a kind of modal talk, as in: 'if a different setting were chosen at a distant apparatus, this should not influence the result actually observed on the near-by apparatus', the framework should somehow merge modality with the prohibition on superluminal signalling. An auspicious model of this kind is provided by Belnap's 'branching space-time' (BST) (Belnap [1992]), given that it permits the introduction of outcomes of events (intuitively, possible results of a measurement). We need therefore a framework capable of merging Belnap's 'branching space-time' with his 'outcomes in branching time'. The triggering impulse for this enterprise comes from Belnap's remark: 'Outcomes in branching space-time: definition unclear, therefore no results' (Belnap [1995]). Our second objective is to show how this framework can be used to analyze common causation in Bell-type setups. There is a known argument to the effect that in a rather special branching space-time, no common causes can exist for the correlations in the three particle Mermin's ([1990a] and [1990b]) version of GHZ-Bell theorems (Belnap and Szabó [1996]). Thus, we need to show that this argument carries over to our (general) framework. We will also investigate whether some other quantum setups (see Sections 6–8) allow for common causes, and observe that in our framework there is a common cause for the results of a setup only if these results can be reconstructed by a local contextual hidden variable model. A model of that kind can be thought of as specifying a set of instructions that each particle involved carries with itself, that says how the particle would behave at the measuring apparatus, given that this or that setting is chosen. The instruction is assumed not to be influenced by what setting is chosen at a remote apparatus (nor, indeed, by what setting is chosen at the particle's own apparatus).

An intriguing question is why certain data can be thought of as outcomes of a common cause, whereas other data do not allow such a reconstruction. Rather thaen accepting this as a brute fact about causation, we intend to look into what algebraic constraints a common cause may impose on the relevant structure of outcomes in BST. At first sight, this search might appear to be an exercise in pure logic, without any bearing on our understanding of the physical world. But, although this view of ours might be controversial, it appears to us that more often than not the best explanation one may aim for is algebraic (or 'structural', as it is sometimes called). Thus, we feel that the issues of causality in microphysics (and of the existence of contextual local hidden variable models) will be much clarified once the algebraic structure of outcomes assumed in GHZ-Bell setups has been shown not to support the algebraic constraint imposed by common causation.

With that confession, we invite the reader in the logical part of the paper.

2 The algebra of outcomes in branching space-time

Our aim is to introduce certain structures in which we will identify objects similar to possible worlds of the familiar Kripke semantics. Our "possible worlds" will differ from the standard in that they will (and not just 'might') possess some internal structure which we will want to think as a world-history, i.e., a maximal possible collection of events a world may encompass. Technically, it boils down to the requirement that possible words be some specific (maximal upward directed, let us say in advance) subsets of a base set W. We will call our possible worlds simply histories. Thus, we will work with structures $\mathbf{W} = \langle W; \leq \rangle$ such that W is a non-empty set, partially ordered by $\leq$. In accord with the intuition that we have just spelled out, we might think of W as a totality of spatiotemporal points, where the point is not merely a spatiotemporal place that can be filled in one way or another, but rather a concrete particular, that is, a point together with whatever there is in it. The relation $x \leq y$ is interpreted as meaning that x is in the backward light cone of y. Since we assume here that the point is in its own backward light cone, $\leq$ is reflexive. It is also transitive and, barring some pathologies, antisymmetric. Accordingly, $\leq$, interpreted as above, is indeed a partial ordering. In a familiar way, we have: $x < y$ iff $x \leq y$ and $x \neq y$. This relation, $<$, captures a prerequisite of subluminal causation rather than causation itself: $x < y$ says that x can causally influence y rather than that x causes y.

Our structures will be required to satisfy certain additional postulates, meant to remain, as long as possible, in agreement with Belnap and probably also with a kind of physical intuition.

Before we state the postulates, it will be convenient to define a couple of auxiliary notions.

DEFINITION 1 (OF COMPATIBILITY)
We say that $x, y \in W$ are upward compatible iff there is a $z \in W$ with $z \geq x$ and $z \geq y$. We say that $x, y \in W$ are downward compatible iff there is a $z \in W$ with $z \leq x$ and $z \leq y$. In standard mathematical usage sets composed entirely of upward (downward) compatible elements are called upward (downward) directed.

We write $x \perp y$ to indicate that x and y are *not* upward compatible, and we refer to such elements as *orthogonal.* For downward incompatibility we do not use any special symbol. It seems likely that any two points are in fact downward compatible (think of the big bang, for instance), but we have chosen not to decide this issue either way.

Next, following Belnap, we introduce some special subsets of W, that we have already alluded to. These we call 'histories'. Intuitively, a history is to represent a possible course of events.

DEFINITION 2 (OF A HISTORY)
A subset h of W is a history iff h is an upward directed subset of W and for all upward directed $h' \subseteq W$ we have: $h' \supseteq h$ implies $h' = h$. In other words: histories are maximal upward directed subsets of W. We denote the set of all histories by $\mathcal{H}$.

Note that with this concept of histories we can accommodate two visions of possible worlds, the vision of branching time where every two worlds (histories) have a common initial segment and the Lewisian vision, which forbids any two words to have a common segment. The definition also assures that orthogonal points are such that one belongs to one history and the other to another. Our motivation for this concept of history derives from the meaning of the fork characterized by $x < z$, $y < z$ and neither $x \leq y$ nor $y \leq x$. This fork can only be interpreted as saying that space-like separated points x and y are located in the backward light cone of z. A modal interpretation of this fork, which says that two points belonging to different histories causally influence some third point, does not seem to have much sense. We will briefly return to this issue in the next section.

Since a course of events may go one way or another, histories split. To exhibit how we think this splitting is effected, let us consider a photon that is approaching a surface of a translucent medium. After hitting the surface, the photon can be either refracted or reflected. Accordingly, we have the history (histories) in which the photon is refracted and the history (histories) in which it it reflected. With some idealization, the histories split in the point of the photon hitting the surface of the medium. Again, this splitting point is not merely a spatiotemporal place, but is thought of as containing all the physical characteristic of that place.

Without yet knowing whether any exist, we may, thus, define the sets of *splitting points* both for points and for histories.

DEFINITION 3 (OF SPLITTING POINTS FOR POINTS)
For any two orthogonal points $x, y \in W$, we define $C(x, y) \subseteq W$ by putting $z \in C(x, y)$ iff z is a maximal element in $\{z \in W : z \leq x \;\&\; z \leq y\}$. If x and y are not orthogonal, we put $C(x, y) = \emptyset$.

DEFINITION 4 (OF SPLITTING POINTS FOR HISTORIES)
For histories $h_1, h_2 \subseteq W$, we define $C(h_1, h_2) \subseteq W$ by putting $z \in C(h_1, h_2)$ iff z is a maximal element in $h_1 \cap h_2$.

We will call $C(x, y)$ the set of *splitting points* for x and y, and analogously for histories. Nothing so far ensures that even for a pair of orthogonal points x and y, $C(x, y)$ is non-empty. For an example of that consider the structure $\mathbf{W_0} = \langle \omega \cup \{a, b\}; \leq \rangle$, where $\leq$ is the natural ordering on the set ω of all natural numbers, and $\forall_{n\in\omega}\ a \geq n \ \&\ b \geq n,\ a \neq b$. Here a and b are orthogonal, but $C(a, b) = \emptyset$.

Even if $C(x, y)$ is non-empty, there might be a path leading to x or y and bypassing $C(x, y)$. Consider the previous structure enriched by one more point c, which is below a and b and above (say) zero, but is incomparable with any other natural number, i.e., the structure $\mathbf{W_1} = \langle \omega \cup \{a, b, c\}; \leq \rangle$, where $\leq$ is the natural ordering on ω, and $\forall_{n\in\omega}\ a \geq n \ \&\ b \geq n,\ a \neq b$, and moreover $c \geq 0$, $c \leq a$, $c \leq b$. Compare Figure 1.

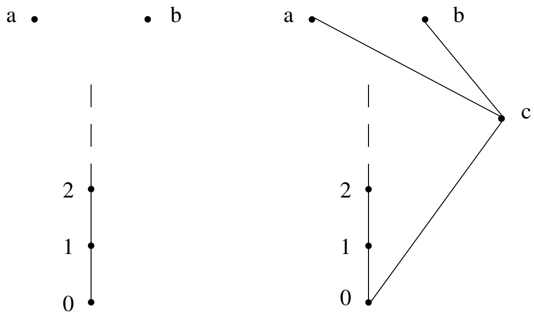

Figure 1: Pathological structures that motivate requirement $C1$

Now we can state the two conditions we place on $\leq$.

(C1) For any $x, y, z \in W$, if $x \perp y$ and $z \leq x$, $z \leq y$, then there is some $t \in C(x, y)$ with $t \geq z$.

(C2) For any $x, y, z, t \in W$, if $x \geq z$ and $y \geq t$, then $C(x, y) \supseteq C(z, t)$.

The first says, that for any orthogonal pair of points (1) the set of their splitting points is non-empty, and (2) any element below them both is also below some splitting point. Thus, it excludes "pathologies" of Figure 1 and brings about the desired effect that any two points a and b belong to a common history h iff there are no splitting points for them. This condition will be frequently made use of, in the form of passing from $C(a, b) = \emptyset$ to: there is a c with $a \leq c$ and $b \leq c$. Without (C1) the left-hand side of Figure 1 would provide a counterexample.

The second says, that splitting points are retained along histories, that is, if two points belong to different courses of events, i.e., the set of their splitting points is non-empty, and we move further on along one or the other course, then we still have the same set of splitting points and, maybe, some more.

Now we define a bunch of notions that are crucial to our enterprise.

DEFINITION 5 (OF PRECEDENCE)
For $E, F \subseteq W$, $x \in W$ (1) $E \prec x$ iff $\forall_{e \in E}\ e < x$; (2) $E \prec F$ iff $\forall_{x \in F}\ E \prec x$.

DEFINITION 6 (OF RELATIVE ORTHOGONALITY)
Elements x, y of W are orthogonal relative to E, written $x \perp_E y$, iff $E \prec x$, $E \prec y$ and $C(x, y) \cap E \neq \emptyset$.

DEFINITION 7 (OF ORTHOGONAL COMPLEMENT)
For $F \subseteq W$, the orthogonal complement of F relative to E is the set $F^{\perp_E}$ such that $x \in F^{\perp_E}$ iff $\forall_{y \in F}\ x \perp_E y$.

DEFINITION 8 (OF OUTCOME)
A subset F of W is an outcome of E iff $F = F^{\perp_E \perp_E}$

Our reason for adopting this definition is that it satisfies some intuitive requirement for outcomes. The outcome of E is preceded by E and is located as close as possible to E. And, what the outcomes of E look like crucially depends on whether and, if so, how many, histories split in E.

Some of the properties of $^{\perp_E}$ are gathered in the lemma below. Let us note on the margin, that Lemma 2 holds for just any reflexive and symmetric relation $\not\perp$ (with the definition of outcome stated in terms of the relation $\perp$, of course).

LEMMA 1
For X, Y, E subsets of W with $E \prec X$, $E \prec Y$, the following hold:

(i) $X \subseteq X^{\perp_E \perp_E}$;

(ii) $X^{\perp_E} = X^{\perp_E \perp_E \perp_E}$;

(iii) $X \subseteq Y$ *implies* $Y^{\perp_E} \subseteq X^{\perp_E}$;

(iv) $X^{\perp_E} = \bigcap \{\{x\}^{\perp_E} : x \in X\}$.

PROOF. For (i), let $x \in X$. Suppose $x \notin X^{\perp_E \perp_E}$. Then, by Definition 7, for some $y \in X^{\perp_E}$ we have $y \not\perp_E x$. This, by Definition 6, unwinds as $C(y, x) \cap E \neq \emptyset$. Since y is in $X^{\perp_E}$, we get $\forall_{z \in X} z \perp_E y$. In particular, $x \perp_E y$. Contradiction.

Once (i) is shown, for (ii) it suffices to prove $X^{\perp_E} \supseteq X^{\perp_E \perp_E \perp_E}$. Let $x \in X^{\perp_E \perp_E \perp_E}$, but $x \notin X^{\perp_E}$. Then, for some $y \in X^{\perp_E}$ we have $y \not\perp_E x$. On the other hand, since $x \in X^{\perp_E \perp_E \perp_E}$, we have $z \perp_E x$, for all $z \in X^{\perp_E \perp_E}$. As by (i) we have $X \subseteq X^{\perp_E \perp_E}$ we have in particular that $y \perp_E x$. Contradiction.

To prove (iii) assume $X \subseteq Y$, $y \in Y^{\perp_E}$, $y \notin X^{\perp_E}$. Then, for some $z \in X$ we have $y \not\perp_E z$. But, by $X \subseteq Y$, z belongs to Y as well. So $\exists_{z \in Y}\ y \not\perp_E z$. A contradiction with $y \in Y^{\perp_E}$.

For (iv), take any $x \in X$ and notice that since $x \in \{x\}$, we have $\{x\}^{\perp_E} \supseteq X^{\perp_E}$ by (iii). Thus, $\bigcap\{\{x\}^{\perp_E} : x \in X\} \supseteq X^{\perp_E}$. On the other hand, if $y \notin \bigcap\{\{x\}^{\perp_E} : x \in X\}$, then $y \notin \{x\}^{\perp_E}$, for some $x \in X$. This means $y \not\perp_E x$, for some $x \in X$, and thus, $y \notin X^{\perp_E}$. This finishes the proof of the other inclusion.

□

COROLLARY 1

(i) *The operation ${}^{\perp_E\perp_E}$ is an algebraic closure operator on the set of those subsets of W that E precedes. Its closed sets are precisely the outcomes of E.*

(ii) *The set of all outcomes of E is a complete lattice with the smallest element $\emptyset$, the greatest element $\Omega_E = \{x \in W : E \prec x\}$, and the lattice operations given by: $F \wedge G = F \cap G$, $F \vee G = (F \cup G)^{\perp_E\perp_E}$.*

PROOF. Recall that an algebraic closure operator is an operator which, in addition to the three usual closure conditions (points (a), (b), (c) below), satisfies also the compactness (or, finitariness) condition saying that the closure of a set is the union of the closures of all its finite subsets (point (d) below). Thus, for (i) we need to check whether:

(a) $X \subseteq X^{\perp_E\perp_E}$, but this is Lemma 1(i);

(b) $X^{\perp_E\perp_E} = X^{\perp_E\perp_E\perp_E\perp_E}$, which follows immediately from Lemma 1(ii);

(c) $X \subseteq Y$ implies $X^{\perp_E\perp_E} \subseteq Y^{\perp_E\perp_E}$, which follows by Lemma 1(iii);

(d) $X^{\perp_E\perp_E} = \bigcup\{Z^{\perp_E\perp_E} : Z \subseteq_{\text{fin}} X\}$, which is the only fact requiring some proof. In fact we will prove that: $\bigcup\{\{x\}^{\perp_E\perp_E} : x \in X\} = X^{\perp_E\perp_E}$.

Here is the argument: Obviously, $\{x\} \subseteq X$ for all $x \in X$, and so $\{x\}^{\perp_E\perp_E} \subseteq X^{\perp_E\perp_E}$. Thus, $\bigcup\{\{x\}^{\perp_E\perp_E} : x \in X\} \subseteq X^{\perp_E\perp_E}$. We are going to establish the reverse inclusion.

Suppose (1): $z \in X^{\perp_E\perp_E}$, but (2) $z \notin \bigcup\{\{x\}^{\perp_E\perp_E} : x \in X\}$.
By definition, (1) is equivalent to:

$$\forall_{y \in X^{\perp_E}}\ z \perp_E y,$$

and further to:

$$\forall_{y\in\Omega}((\forall_{x\in X}\ x \perp_E y) \Rightarrow z \perp_E y),$$

which in turn, can be rewritten in its equivalent prenex form:

$$\forall_{y\in\Omega}\exists_{x\in X}\ (x \perp_E y \Rightarrow z \perp_E y). \tag{†}$$

By definition again, (2) is equivalent to:

$$\forall_{x\in X}\ z \notin \{x\}^{\perp_E\perp_E},$$

and further to:

$$\forall_{x \in X} \exists_{y \in \{x\}^{\perp_E}} \; z \not\perp_E y,$$

and yet further to:

$$\forall_{x \in X} \exists_{y \in \Omega} \; x \perp_E y \;\&\; z \not\perp_E y. \tag{‡}$$

Now pick out a pair x_0, y_0 of elements of Ω satisfying (‡) above. We thus have $z \not\perp_E y_0$. On the other hand, x_0, y_0 satisfy the antecedent of (†), so we must have $z \perp_E y_0$, which is a blatant contradiction. Thus, we obtain that $\bigcup\{\{x\}^{\perp_E \perp_E} : x \in X\} = X^{\perp_E \perp_E}$, and $\{x\}$ is clearly a finite subset of X.

For (ii) we need only to show that the lattice join is in fact what we claim it is, as the rest follows by the well-known fact of lattice theory that the collection of all closed sets under an algebraic closure operator is a complete algebraic lattice. In fact, we do not need the algebraicity of the resulting lattice to state (ii), we note it here only for the record.

Thus, we will prove that $(F \cup G)^{\perp_E \perp_E}$ is the smallest outcome containing $F \cup G$. By point (i), all intersections (even infinite) of closed sets (here called outcomes) are themselves closed, so the smallest outcome containing $F \cup G$ is $\bigcap\{X \supseteq F \cup G : X \text{ an outcome}\}$. We have to show that this equals $(F \cup G)^{\perp_E \perp_E}$. First notice, that by (ii), $F \cup G \subseteq (F \cup G)^{\perp_E \perp_E}$, and by (iii) the latter is an outcome, so $\bigcap\{X \supseteq F \cup G : X \text{ an outcome}\} \subseteq (F \cup G)^{\perp_E \perp_E}$. For the other inclusion, take any $x \in (F \cup G)^{\perp_E \perp_E}$ and suppose $x \notin X$, for some outcome X containing $F \cup G$. Thus, for some $y \in X^{\perp_E}$ we have $y \not\perp_E x$. On the other hand, for all $t \in (F \cup G)^{\perp_E}$, $t \perp_E x$, by the fact that $x \in (F \cup G)^{\perp_E \perp_E}$. Now, as $X \supseteq F \cup G$, we have $X^{\perp_E} \subseteq (F \cup G)^{\perp_E}$ and hence $y \in (F \cup G)^{\perp_E}$, yet $y \not\perp_E x$. Contradiction.

□

LEMMA 2

The lattice of outcomes is an ortholattice.

PROOF. Firstly, we show that it is orthocomplemented. That $F^{\perp_E \perp_E} = F$ follows trivially by the fact that F is an outcome. Consider $F \cap F^{\perp_E}$. If this is non-empty, then some x belongs to both F and $F^{\perp_E}$. But then, by the definition of $^{\perp_E}$, we would have that, in particular, $x \perp_E x$ which is a contradiction. Thus, $F \cap F^{\perp_E} = \emptyset$.

Now, consider $F \vee F^{\perp_E}$. Suppose this is not equal to Ω_E, so that there is an $x \in \Omega_E$ with $x \notin (F \cup F^{\perp_E})^{\perp_E \perp_E}$. Thus, there is a $z \in (F \cup F^{\perp_E})^{\perp_E}$ with $x \not\perp_E z$. Further: $\forall_{y \in F \cup F^{\perp_E}} \; z \perp_E y$. From this we obtain, on the one hand $\forall_{y \in F} \; z \perp_E y$, and hence $z \in F^{\perp_E}$. On the other hand, $\forall_{y \in F^{\perp_E}} \; z \perp_E y$, from which it follows that $z \in F^{\perp_E \perp_E} = F$. But the intersection of F and $F^{\perp_E}$ is empty by previous argument. Again, we have arrived at a contradiction. Therefore, $F \vee F^{\perp_E} = \Omega_E$.

Secondly, we show that de Morgan laws hold. It suffices to show that $(F \vee G)^{\perp_E} = F^{\perp_E} \cap G^{\perp_E}$. By previous facts, $(F \vee G)^{\perp_E} = (F \cup G)^{\perp_E \perp_E \perp_E} = (F \cup G)^{\perp_E}$. Then, $x \in (F \cup G)^{\perp_E}$ iff $\forall_{z \in F \cup G}\ x \perp_E z$ iff $\forall_{z \in F}\ x \perp_E z$ and $\forall_{z \in G}\ x \perp_E z$ iff $x \in F^{\perp_E}$ and $x \in G^{\perp_E}$ iff $x \in F^{\perp_E} \cap G^{\perp_E}$ as required.

□

Note that all the proofs up to here have not made any use of our condition (C2), although (C1) has been, implicitly yet frequently, employed. It is from the next lemma that (C2) will begin to do its work. Technically, it means that we have overshot the target at places, e.g., defining $F \vee G$ to be $(F \cup G)^{\perp_E \perp_E}$, for it will soon turn out — with the help of (C2) — that the set-theoretical union of outcomes is itself an outcome. Yet, the authors believe they have sacrificed such gains in simplicity as this might have brought, for the sake of greater transparency of the roles of (C1) and (C2).

LEMMA 3
Outcomes are upward closed, i.e., if $x \leq y$, $x \in F$, then $y \in F$.

PROOF. Suppose $x \leq y$, $x \in F$. Thus, obviously, $E \prec y$. Then, since $x \in F^{\perp_E \perp_E}$, we have $\forall_{z \in F^{\perp_E}}\ x \perp_E z$. By definition, this is $C(x, z) \cap E \neq \emptyset$. Now since $y \geq x$, by (C2) we get $C(y, z) \supseteq C(x, z)$. Thus, $C(y, z) \cap E \neq \emptyset$, for all $z \in F^{\perp_E}$, which, by definition, yields $y \in F^{\perp_E \perp_E} = F$.

□

Note an implication of Lemma 3: the outcome continues as long as histories it involves continue. An outcome of a coin toss does not merely contain one or the other way the coin can land, but also whatever may (in the causal sense) happen after the coin landed in a given way.

LEMMA 4
For any $x, y \in W$, and $E \subseteq W$, if $E \prec x$, $E \prec y$, and $y \notin \{x\}^{\perp_E \perp_E}$, then $y \in \{x\}^{\perp_E}$.

PROOF. Let x, y, E be as in the assumption, and suppose $y \notin \{x\}^{\perp_E}$, that is $C(x, y) \cap E = \emptyset$. We have:

$$\forall_{z \in \{x\}^{\perp_E}}\ C(x, z) \cap E \neq \emptyset,$$

$$\exists_{t \in \{x\}^{\perp_E}}\ C(y, t) \cap E = \emptyset.$$

So let's pick a t_0 out of $\{x\}^{\perp_E}$; we get: $C(x, t_0) \cap E \neq \emptyset$ and $C(y, t_0) \cap E = \emptyset$.

Case 1. Suppose first that $C(x, y) \neq \emptyset$. Therefore,

$$\forall_{e \in E} \exists_{s \in C(x,y)}\ e < s. \tag{$*$}$$

For otherwise, we would have some $e \in E$ not strictly smaller than any minimal element below both x and y, and since $e \leq x$, $e \leq y$, e itself would belong to $C(x, y)$ contradicting the fact that $C(x, y) \cap E = \emptyset$.

Now we split the proof into two further cases:

Case 1.1. $C(y, t_0) = \emptyset$. Then, there is a $w \in W$ with $w \geq y$, $w \geq t_0$; thus, $C(w, x) \supseteq C(y, x) \cup C(t_0, x)$. Hence, $C(w, x) \cap E \neq \emptyset$. But, on the other hand, if $e \in C(t_0, x) \cap E \subseteq C(w, x) \cap E$, then, by $(*)$, there is an $s \in C(x, y)$ with $e < s$. Thus, $e \notin C(w, x)$ which is a contradiction.

Case 1.2. $C(y, t_0) \neq \emptyset$. Precisely as in $(*)$, this yields,

$$\forall_{e \in E} \exists_{r \in C(y, t_0)}\; e < r.$$

Take any $e \in E$ and pick out an $r \in C(y, t_0)$ with $e < r$. We will show that $e \notin C(x, t_0)$. To this end consider $C(x, r)$. We have: $C(x, r) \subseteq C(x, y) \cap C(x, t_0)$. Thus, $C(x, r) \cap E = \emptyset$, in particular $e \notin C(x, r)$. If $e \in C(x, t_0)$, then there is some element q of $C(x, r)$ with $q > e$, for if it were otherwise e would belong to $C(x, r)$. Now, $q \leq x$ and $q \leq r \leq t_0$. Thus, $e \notin C(x, t_0)$, as needed. Since our e was arbitrary, we can conclude that $\forall_{e \in E}\; e \notin C(x, t_0)$, i.e., $C(x, t_0) \cap E = \emptyset$, contradicting the assumption.

Case 2. If $C(x, y) = \emptyset$, then we have some $z \in W$ with $z \geq x$, $z \geq y$. The argument splits again. If $C(y, t_0) \neq \emptyset$, then we argue as in Case 1.1, simply interchanging $C(x, y)$ with $C(y, t_0)$. If $C(y, t_0) = \emptyset$, then there is a v above y and t_0. Now, if $C(z, v) = \emptyset$, then there is something above x and t_0 as well. This cannot happen, as t_0 is a member of $\{x\}^{\perp_E}$. Thus, $C(z, v)$ is non-empty and, by (C2), $C(z, v) \supseteq C(x, t_0)$. In particular, $C(z, v) \supseteq C(x, t_0) \cap E$. Members of the latter, however, cannot belong to $C(z, v)$. For both z and v are above y, and since $E \prec y$, anything that belonged to $C(x, t_0) \cap E$ would not be maximal below z and v. This is a contradiction again, and with it the whole proof ends.

□

LEMMA 5

The lattice of outcomes is a boolean algebra. Moreover, the boolean algebra of outcomes is complete and atomic.

PROOF. For the first part, it suffices to prove that the complement is unique, that is, for any outcomes F, G of E, if we have: $F \vee G = \Omega_E$ and $F \cap G = \emptyset$, then $G = F^{\perp_E}$. To prove this, it is, in turn, sufficient to show that if $E \prec x$ and $x \notin F$, then $x \in F^{\perp_E}$.

If $x \notin F$ then, by Corollary 1(i) (point (d) in its proof), $x \notin \{y\}^{\perp_E \perp_E}$ for any $y \in F$. By Lemma 3, we get $x \in \{y\}^{\perp_E}$. Hence, $x \in \bigcap\{\{y\}^{\perp_E} : y \in F\}$. This set, by Lemma 1(v), is precisely $F^{\perp_E}$.

Before we set out to prove the "moreover" part, let us spell out what atomic outcomes are. The definition is completely standard, and would not perhaps be worth the name, were it not for its applications to come.

DEFINITION 9 (OF ATOMIC OUTCOMES)
e is an atomic outcome of $E \subset W$ iff (1) e is a non-empty outcome of E and (2) there is no non-empty outcome u of E such that $u \subset e$.

Coming back to the proof of Lemma (5), we will now show that whenever there is an element z with $E \prec z$, then $\{z\}^{\perp_E \perp_E}$ is an atomic outcome of E.

Notice first that the set $\{z\}^{\perp_E \perp_E}$ cannot be empty. For if it were, then, $\{z\}^{\perp_E} = \Omega_E$, and thus, by definition, for all x with $E \prec x$ we would have: $x \perp_E z$. In particular, $z \perp_E z$, which is a contradiction.

Therefore, $\{z\}^{\perp_E \perp_E}$ is non-empty. By Corollary 1 (b), $\{z\}^{\perp_E \perp_E}$ is an outcome of E. Suppose now that there is a non-empty outcome u of E such that $u \subset \{z\}^{\perp_E \perp_E}$. If $z \in u$, then as $\{z\} \subseteq u$, we get $\{z\}^{\perp_E \perp_E} \subseteq u^{\perp_E \perp_E} = u$. This is impossible for we have assumed that the inclusion $u \subset \{z\}^{\perp_E \perp_E}$ is proper. Thus, $z \notin u = u^{\perp_E \perp_E}$. Let y be any element from u. Then, $\{y\}^{\perp_E \perp_E} \subseteq u^{\perp_E \perp_E} = u$. Thus, $z \notin \{y\}^{\perp_E \perp_E}$. By Lemma 4 we get, $z \in \{y\}^{\perp_E}$. Since the argument can be repeated for any $y \in u$, we obtain that $z \in \bigcap\{\{y\}^{\perp_E} : y \in u\}$. By Lemma 1(vi), we get that $z \in u^{\perp_E}$. Thus, $\{z\}^{\perp_E} \supseteq u^{\perp_E \perp_E} = u$, and, by boolean algebra, we obtain that $u \subseteq \{z\}^{\perp_E \perp_E} \cap \{z\}^{\perp_E} = \emptyset$. Contradiction.

Now, to conclude that the algebra of outcomes is indeed atomic we have to show that every non-zero element has an atom below, which translates into the language of outcomes as: Every non-empty outcome contains an atomic outcome. The latter, however, is trivially true, by Corollary 1 (part (d) in its proof).

The last thing we need is that the algebra of outcomes is complete. This is Corollary 1(ii).

The proof is now complete, provided that there indeed is an element z with $E \prec z$, as we have assumed. If no such a z exists, and $E \neq \emptyset$, then, by Definition 8, E has only one outcome, the empty set. Thus, the algebra of outcomes has only one element (i.e., is trivial, in algebraic parlance), and such an algebra is atomic, for it has no non-zero elements.

$\square$

One consequence of the above is that the set-theoretical union of two outcomes is again an outcome, a feature that is not present in Belnap's outcomes in branching time. In spite of that, the boolean algebra of outcomes need not be the usual set algebra (the complement is, in "standard cases" different), although the latter is a particular case of the former. Namely, when we take the family $\mathcal{S}(X) = \big\{\{x\} : x \in X\big\} \cup \{\emptyset\}$ ordered by inclusion, and then consider the outcomes of $\{\emptyset\}$ in the structure $\langle \mathcal{S}(X), \subseteq \rangle$ we get the usual power set algebra, i.e., the structure $\langle \wp(\mathcal{S}(X)), \subseteq \rangle$.

Thus, by Stone representation, the following converse relation also holds.

LEMMA 6
For every complete atomic boolean algebra $\mathcal{B}$ there is a world W and an event $E \subset W$ such that the algebra of outcomes of E is isomorphic to $\mathcal{B}$.

Finally, let us take a closer look at the outcomes of some specific subsets of W. Since every outcome of E must be preceded by E, for any E containing a pair of orthogonal points, its only outcome is the empty set. The same holds for any subset E of W that is not bounded from above, i.e. the only outcome of E is $\emptyset$. There are, however, subsets of W that, as we will soon see, must have non-empty outcomes. We will dub these subsets 'events' and define them as follows:

DEFINITION 10 (OF EVENTS)
We say that $E \subset W$ is an event if $E \neq \emptyset$ and $\exists_{x \in W}\ E \prec x$.

LEMMA 7
$E \subset W$ is an event iff E has a non-empty outcome.

PROOF. To prove the implication to the right, notice that the set $o = \{x : E \prec x\}$ is non-empty, as E is an event. And, since $o^{\perp_E}$ consists of exactly those points of W that E precedes and are orthogonal (relative to E) to *all* points of o, $o^{\perp_E}$ must be empty. Accordingly, $o^{\perp_E \perp_E} = \{x : E \prec x\} = o$, which shows that o is an outcome of E.
For the converse, let o be a non-empty outcome of E. Thus, we have $\forall_{x \in o}\ E \prec x$. And, since o is non-empty, this proves that E is an event.

□

The next lemma is a direct consequence of Lemma 5.

LEMMA 8
Every event $E \subset W$ has an atomic outcome.

REMARK 2.1 *We would like to end this section with two easy exercises, and one more serious question. The exercises are these:*

(1) Most of our lemmas do not make the assumption $\emptyset \neq E \neq W$. Only after events have been defined do we require that. What happens, if we drop this requirement? More specifically, what if $E = \emptyset$? What if $E = W$? Do we get boolean algebras? If we do, describe them. Are they atomic?

(2) After the proof of Lemma 5 we said that one of its consequences is that the set-theoretical union of two outcomes is itself an outcome. Prove this statement. What about a union of any finite number of outcomes? Of an infinite number?

The question is concerned with the role played by the condition (C2), and is rather technical in nature. As we have seen, the structure of "outcomes" we obtain without (C2), is that of a complete orthomodular lattice, and only by virtue of (C2) we make the great leap forward to complete and atomic boolean algebras. Is there any, fairly natural, condition to replace (C2) with in order to obtain a less restrictive structure? For instance, an orthomodular lattice?

3 Physical Interpretation

Our interpretation of branching space-time is essentially that of Belnap.[4] It is outcomes where some differences occur, since we deal with outcomes in branching space-time whereas Belnap's outcomes reside in branching time. Hence, we will briefly rehearse the interpretative issues of the BST and focus our attention on the problems of outcomes.

Branching space-time involves a richer structure of forks than branching time. In the framework of the latter, there are only one kind of forks, call them 'upward', that is forks with $x < y$, $x < z$ and $y \perp z$. Moreover, the upward fork receives there an unequivocal interpretation: from x on, the course of events can go one way or the other, each of these two histories containing a point that is incompatible with a point belonging to the other history. Contrary to this, in BST the upward fork, may receive one of two possible interpretations, and apart from upward forks, there will be downward forks as well (see Figure 2).

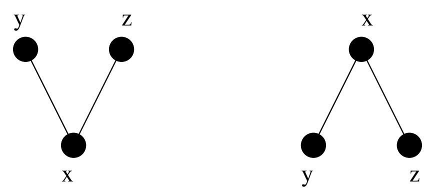

Figure 2: Upward and downward forks in BST; the lines represent the ordering relation $<$.

An upward fork may denote, as in branching time, the splitting of two histories, but, it also may capture a situation with one point located in backward light cones of two space-like separated points. Thus, in the latter case, x, y, and z belong to one history, whereas in the former case they cannot be in one history. The downward fork, however, allows for only one interpretation, that is, two space-like separated points located in the backward light cone of a third point. A robust sense of reality protests against the idea that a course of events can be causally influenced by something that does not belong to this very course of events. Accordingly, if for some incomparable x and y there is a z such that $x < z$ and $y < z$, then x and y belong to a history (or, more generally, to some histories). And *vice versa*, if x and y are in a history, then there is some z, whose past contains both x and y. These intuitions are built in our definition of histories as maximal upward connected subsets of W.

What we called 'event' is any subset of W that is fully contained in a history but definitely shorter than that history. We may think of the event as a chunk of a spatiotemporal history, though this need not be our actual history. Relation $\prec$ of precedence allows us to trace causal influences between events. Some events can be thought of as measuring procedures, but nothing in our framework depends on this identification.

This brings us to outcomes of subsets of W. To arrive at outcomes of E we ask: given that E took place, what are the possible results of its happening? Note first that only events have non-empty outcomes (see Lemma 7). What these outcomes look like depends on whether or not there are histories that split in the event in question. If there are no such histories, that is, E has only an empty intersection with the set of splitting points of any two points of W, then E has only trivial outcomes, i.e., the empty outcome and $\Omega_E = \{x : E \prec x\}$. Things look differently if an intersection mentioned above is not empty. This means that in E some histories split, each of which later on may split again. Thus, to be more precise, these are *bundles* of histories that split in E. Intuitively, an atomic outcome of E consists of all points preceded by E that belong to the smallest possible bundle of histories that splits off in E.

Our outcomes in BST are governed by the requirement (which has no counterpart in Belnap's definition of outcomes in branching time) that the event must precede all its outcomes. This arises from our definition of relative orthogonality

$$x \perp_E y \qquad \text{iff} \qquad E \prec x \;\&\; E \prec y \;\&\; C(x, y) \cap E \neq \emptyset,$$

where we require, in addition to Belnap's definition, that both x and y be preceded by E. Now, outcomes in *branching time* with Belnap's definition of the relative orthogonality are known to form an orthomodular lattice, which appears to be highly welcome because of a possible link with quantum logic. This link, however, seems to be rather elusive as, firstly, the Belnap-style lattices are all made up of boolean algebras glued together at their largest and smallest elements only (these are called $(0, 1)$-pastings in ortholattice-speak), and thus cannot account even for the quantum propositional system for a spin-one particle. Secondly, some of these are finite and non-modular, a phenomenon that has no counterpart in the usual algebraic structures associated with QM (von Neumann algebras). Thus, it is not altogether unwelcome that the family of outcomes in our BST model is a boolean algebra and not only a weaker orthomodular lattice.

Finally, note this implication of our definition of $\perp_E$: For points x and y we say $x \perp_E y$ given that *some* splitting points (rather than *all* splitting points) of x and y are in E. This claim means that an outcome of E can also be an outcome of some other event. Accordingly, if a subset o of W is an outcome of event E, this means that E can causally influence o, but there may be other events that do so as well.

4 Common Causes in BST

In this section we will introduce the concept of non-stochastic common cause. We call this kind of common cause 'non-stochastic' because in this paper we do not assign probabilities to outcomes and accordingly our common causes are designed to account for the non-existence of certain outcomes rather than some peculiar probability distribution of outcomes. We begin with some definitions:

DEFINITION 11 (OF CONSISTENT SUBSETS OF W)
$E_1 \subset W$ is consistent with $E_2 \subset W$ if there is a history that intersects both E_1 and E_2; in symbols

$$\exists_h \, (h \cap E_1 \neq \emptyset \; \& \; h \cap E_2 \neq \emptyset).$$

We say that E_1 is *inconsistent* with E_2 iff it is not the case that E_1 is consistent with E_2.

DEFINITION 12 (OF SPACE-LIKE EVENTS)
The set $\{E_1, E_2, E_3, \ldots, E_n\}$ of events is space-like only if

- *$\bigcup_i^n E_i$ is an event;*
- *E_i does not overlap with any outcome of E_j, i.e. for all outcomes o_{E_j} of E_j,*

$$E_i \cap o_{E_j} = \emptyset.$$

The second clause of the above definition is perhaps worth commenting upon. Our framework of BST ensures that every point that can be causally influenced by E_i is in some outcome of E_i. Accordingly, to say that there is no overlap between E_k and any outcome of E_i, $k \neq i$, simply means that E_k cannot be influenced by E_i.

Let us now consider under what circumstances the search for a common cause normally begins. Firstly, there must be an event of the form $\bigcup_i E_i$, where $\{E_1, E_2, \ldots, E_n\}$ is space-like. As an illustration for $n = 2$, one may think of the EPR correlations and identify event E_i with the event of choosing a direction of polarization at location i. Accordingly, $\bigcup_i E_i$, $i = 1, 2$ consists of setting the directions at the two locations, this being done in such a way that the constituent events are space-like. Since each E_i and $\bigcup_i E_i$ are events, they have non-empty sets of outcomes. The set of outcomes of E_i consists of the outcome containing the observation of $+$ at location i and the outcome containing the observation of $-$ at location i and the combinations of these two. As for $\bigcup_i E_i$, its outcomes are: the outcome containing two pluses $++$, the outcome containing two minuses $--$ and the outcomes that are combinations of these two outcomes. Now, contrary to our expectations based on an intuitive combinatorics, an outcome of $\bigcup_i E_i$ is missing, namely the outcome containing $+-$. This is because outcome $+$ of E_1 does not overlap with outcome $-$ of E_2 (see Figure 3). More generally, if some outcome of $\bigcup_i E_i$ is missing, there must be outcomes $e_1, e_2, \dots, e_n$ of events $E_1, E_2, \dots, E_n$, respectively, such that $\bigcap_i e_i = \emptyset$.

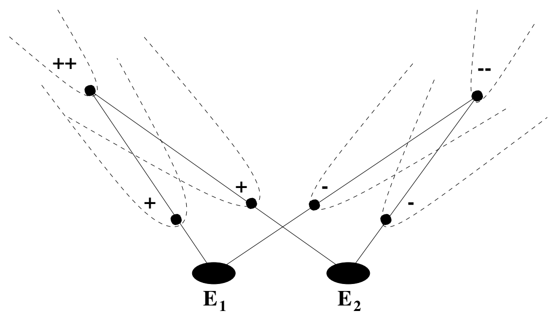

Figure 3: Each event E_i, $i = 1, 2$ has two atomic outcomes that contain $+$ and $-$, respectively. Atomic outcomes of event $E_1 \cup E_2$ start with either $--$ or $++$. There is no outcome of $E_1 \cup E_2$ containing $-+$ (or $+-$). The straight lines represent the relation of precedence, $\prec$.

DEFINITION 13 (OF COMMON CAUSE)
Let $\{E_1, E_2, \dots, E_n\}$ be a space-like set in W and for some outcomes $e_1, \dots, e_n$ of events $E_1, \dots, E_n$, respectively, $\bigcap_i e_i = \emptyset$.
A common cause CC that accounts for $\bigcap_i e_i = \emptyset$ is an event that satisfies:

- $\forall_i\ CC \prec e_i$ *[causal precedence];*
- *Every history that contains CC, contains $\bigcup_i E_i$ as well [conservation];*
- *Every atomic outcome of CC is inconsistent with at least one e_i, $i = 1, \dots, n$ [screening-off].*

To comment on the rationale of the subsequent clauses above, the first is a natural requirement that a common cause precedes its putative effects. As for the second clause, note that we neither require that a common cause precede E_i nor that it precede $\bigcup_i E_i$. However, since we are concerned with outcomes of event $\bigcup_i E_i$, and, thus, we assume that this event has happened, we bar such common causes that would permit one to avoid the event in question. Hence, if CC is in a history, $\bigcup_i E_i$ is in it as well. Finally, the third clause says more about what a common cause should look like and gives a minimal answer to why $\bigcap_i e_i = \emptyset$. In accord with the first clause, CC precedes every e_i, but it is also necessary that in CC bundles of histories diverge, there being at least two such bundles, each going through different outcome e_i. Otherwise, an atomic outcome of CC could be consistent with all e_j. We consider here only atomic outcomes because they represent minimal alternative courses of events. Another reason is that some non-atomic outcomes of CC, e.g., outcome Ω_{CC} of CC, are consistent with all e_i. Now, screening-off says that no matter which atomic outcome of CC is actualized, it is inconsistent with at least one e_i.

It should be noted that definition given above can be used only in the case of missing outcomes of a *single* event composed from a space-like set. For a proof of a Bell-type theorem, however, we have to resort to more than one pair of polarization settings, each setting being chosen at one of two locations. In what follows, we will view an act of choosing a pair of settings as an event composed from a space-like set and investigate outcomes of it. In the present setup, however, with more than one pair of settings, conservation entails that all the acts of choosing pairs of polarization settings must be in one history, which is absurd. Hence the definition above needs to be modified so as to apply to the search of one common cause that accounts for missing outcomes of more than one event (*common* common cause, as it has been called recently).[5]

DEFINITION 14
Let $F^1 = \{E^1_1, E^1_2, \dots, E^1_n\}, F^2 = \{E^2_1, E^2_2, \dots, E^2_n\}, \dots, F^k = \{E^k_1, E^k_2, \dots, E^k_n\}$ be space-like sets in W and for each $j = 1, \dots, k$ there are outcomes $e^j_1, e^j_2, \dots, e^j_n$ of $E^j_1, E^j_2, \dots, E^j_n$, respectively, such that $\bigcap_i e^j_i = \emptyset$ for all j.

A common cause CC that accounts for the k facts, that $\bigcap_i e^j_i = \emptyset$ ($j = 1, \dots, k$), is an event that satisfies:

- *$\forall_{i,j}\ CC \prec e^j_i$ [causal precedence];*
- *Every history that contains CC, contains $\bigcup_i E^j_i$ for some j [conservation];*
- *For each $j \le k$, every atomic outcome of CC is inconsistent with at least one e^j_i, $i \le n$. [screening-off].*

Conservation has a significant implication that is the content of the following Lemma.

LEMMA 9
Every outcome o_{CC} of CC is consistent with at least one outcome $o_{\bigcup_i E_i^j}$ of $\bigcup_i E_i^j$ for some $j \leq k$.

PROOF. Every outcome of CC is preceded by CC and hence every history that intersects any outcome of CC, contains CC. By Conservation, this history must also contain $\bigcup_i E_i^j$ for some j, and hence has a non-empty intersection with an outcome of $\bigcup_i E_i^j$.

5 Is there a common cause in the Bell-GHZ setup with three particles?

Consider now the Mermin *Gedanken* experiment (Mermin [1990b]), which is, in essence, a non-statistical version of a Bell-type theorem with three particles. Each particle from a given trio flies away from a source towards one of three 'stations', the angular distance between any two stations being 120^o. Now, at each station one of two parameters x and y, call them, 'directions', is selected and the passage of a particle through a station brings about one of two results, say $+$ or $-$. Finally, a triple of such pluses and minuses is registered. To introduce the notation, here is an example:

- x_i and y_i, where $i = 1, 2, 3$, are events of setting, respectively, direction x and y at station i.
- $x_1 \cup x_2 \cup y_3$ is the event consisting of the three events of setting the corresponding directions at the three stations. We will use the abbreviation $E_{x,x,y} \stackrel{\text{df}}{=} x_1 \cup x_2 \cup y_3$ and analogously for other events of this kind;
- $x_i^{\pm}$ ($y_i^{\pm}$) is an outcome of event x_i (y_i) - it starts with result $\pm$ registered at station i, with the direction set at x (y) at the station.
- $\langle x_1^+, x_2^-, y_3^+\rangle$ is an 'observation event' that consists of registering the triple of pluses and minuses: at station 1 result $+$ with the direction set to x and at station 2 result $-$ with the direction set to x and at station 3 result $+$ with the direction set to y. There is a straightforward link between the existence of the observation event and the intersection of outcomes of the corresponding directions setting being non-empty. For instance, there exists observation event $\langle x_1^+, x_2^-, y_3^+\rangle$ if and only if $x_1^+ \cap x_2^- \cap y_3^+ \neq \emptyset$.

Now, we have the following stipulations that are either motivated by actual tests of Bell-type theorems, or derive from quantum theory:

Setup Stipulation 1
Every set:
$\{x_1, x_2, x_3\}$, $\{x_1, x_2, y_3\}$, $\{x_1, y_2, x_3\}$, $\{y_1, x_2, x_3\}$, $\{y_1, y_2, x_3\}$, $\{y_1, x_2, y_3\}$, $\{x_1, y_2, y_3\}$ and $\{y_1, y_2, y_3\}$
is space-like.
This assumption is motivated by Aspect *et al's* test of a Bell-type theorem, in which the directions at the measuring devices were set only when the particles were sufficiently distant one from another and close enough to the measuring devices (Aspect, Grangier and Roger [1981 and [1982]). Accordingly, a message about the direction set at one wing could not be transmitted with subliminal velocity to a result measured at another wing. In other words, there was no overlap between outcomes of setting a direction at one wing and the event of setting a direction at another wing.

Setup Stipulation 2
There is a link between an observation event, e.g., $\langle x_1^-, x_2^+, x_3^- \rangle$, and the corresponding outcomes, here x_1^-, x_2^+ and x_3^-, of events of direction setting:
The observation event is contained in a history if and only if this history intersects each corresponding outcome of the events of direction setting.
In symbols, for event $\langle x_1^-, x_2^+, x_3^- \rangle$

$$\langle x_1^-, x_2^+, x_3^- \rangle \subset h \text{ iff } h \cap x_1^- \neq \emptyset \,\&\, h \cap x_2^+ \neq \emptyset \,\&\, h \cap x_3^- \neq \emptyset.$$

Setup Stipulation 3
If a history contains x_i (y_i), it must go through either x_i^+ (y_i^+), or x_i^- (y_i^-).

QM Stipulation
QM imposes a constraint on what triples of pluses and minuses can be observed.[6] In our framework this translates into a claim about which observation events exist or, equivalently, which intersections of outcomes of events of direction setting are non-empty.

- $x_1^k \cap x_2^l \cap x_3^m \neq \emptyset$ iff $k \times l \times m = -1$
- $y_1^k \cap y_2^l \cap y_3^m \neq \emptyset$ iff $k \times l \times m = -1$
- intersection of outcomes of this kind but with x and y mixed is non-empty iff $k \times l \times m = 1$

where $k, l, m = \pm 1$.

Idealization
The assumptions above guarantee that every outcome of $E_{*,*,*}$ contains at least one observation event of the appropriate kind and permitted by quantum theory.

To spell it out for $E_{x,x,x}$, every outcome of $E_{x,x,x}$ must contain at least one event $\langle x_1^i, x_2^j, x_3^k \rangle$, where $i \times j \times k = -1$. This still leaves open the possibility that although a history contains event $E_{*,*,*}$, it does not go through any observation event of the corresponding kind. To exclude this, we assume, as an idealization, that every outcome of $E_{*,*,*}$ *starts* with at least one observation event of the corresponding kind. This entails that every history that goes through an outcome of $E_{*,*,*}$ contains an observation event of the appropriate kind and permitted by QM Stipulation.

To spell it out for $E_{x,x,x}$, every history containing $E_{x,x,x}$ must contain at least one event $\langle x_1^i, x_2^j, x_3^k \rangle$, where $i \times j \times k = -1$.

No Conspiracy

The outcome of a common cause that should account for missing observation events can be thought of as a set of instructions carried by a triple of flying particles. The instruction for a given particle says whether $+$ or $-$ would be displayed at the station, given that direction x or y is chosen there. Now, to exclude conspiracy, we assume that the instruction carried by the incoming particle cannot be changed by a swap of direction settings at the apparatus. In other words, the fact that a triple of particles carries with itself a given set of instructions should not prohibit any choice of direction settings at any station. In the framework of BST, this translates into:

Every outcome of common cause CC should be consistent with every x_i and y_i, $i = 1, 2, 3$.

Now, we have come to the central theorem of this section.

THEOREM 1

There is no single event CC that is a common cause for every missing outcome of each of $E_{x,x,x}$, $E_{x,x,y}$, $E_{x,y,x}$ and $E_{x,y,y}$.

PROOF. Lemma 9 implies that every atomic outcome o_{CC} of CC is consistent with an outcome of $E_{x,x,x}$ or an outcome of $E_{x,x,y}$ or an outcome of $E_{x,y,x}$ or an outcome of $E_{x,y,y}$. Since (by causal precedence) $CC \prec x_l^{\pm}$ for $l = 1, \ldots, 3$ and (by QM Stipulation) for some i, j, k such that $x_1^i \cap x_2^j \cap x_3^k \neq \emptyset$, there is an atomic outcome of CC that is consistent with $E_{x,x,x}$. By Idealization and QM Stipulation, there must be history h that intersects o_{CC} and contains event $\langle x_1^i, x_2^j, x_3^k \rangle$, where $i \times j \times k = -1$. By Setup Stipulation 2, h must go through each outcome x_1^i, x_2^j and x_3^k and hence

- o_{CC} is consistent with each x_1^i, x_2^j and x_3^k

Consider next $E_{x,x,y}$ and take set $x_1^i \cap x_2^j \cap y_3^k$, which is missing by QM Stipulation. By screening-off and the fact above

- o_{CC} is inconsistent with y_3^k

However, by the requirement of no conspiracy, o_{CC} must be consistent with y_3 and hence, by Setup Stipulation 3,

- o_{CC} is consistent with y_3^{-k}

In the next step, consider event $E_{x,y,y}$ and set $x_1^i \cap y_2^{-j} \cap y_3^{-k}$, which is missing (by QM Stipulations). By screening-off and consistencies established above, we have

- o_{CC} is inconsistent with y_2^{-j}

Finally, let us focus attention on $E_{x,y,x}$ and $x_1^i \cap y_2^j \cap x_3^k = \emptyset$. Since o_{CC} is consistent with x_1^i and x_3^k, by screening-off it must be that

- o_{CC} is inconsistent with y_2^j.

However, inconsistency of o_{CC} with both y_2^{-j} and y_2^j implies (by Setup Stipulations 2 and 3) inconsistency of o_{CC} with y_2, and this contradicts the requirement of no conspiracy.

□

Note the power of the requirement of no conspiracy. Indeed, without this condition it is easy to construct a common cause for the setup considered. It suffices to postulate an event that precedes every $x_i^{\pm}$ and $y_i^{\pm}$, $i = 1, 2, 3$ for which also Conservation holds.

6 Is there a common cause in the GHZ-Bell setup with four particles?

Consider now GHZ-Bell argument for four particles[7], with four stations, the directions being as follows:

at station 1: x_1, y_1 and z_1;
at station 2: x_2;
at station 3: x_3 and y_3;
at station 4: x_4 and y_4.

Like in the previous section, we have space-like sets of the form $\{x_1, x_2, y_3, y_4\}$, their unions, like $E_{x,x,y,y} = x_1 \cup x_2 \cup y_3 \cup y_4$, and the observation events, say, $\langle x_1^+, x_2^-, y_3^+, y_4^- \rangle$, that is, quadruples of pluses and minuses.

We have also requirements that look exactly like those of Setup Stipulations 1 - 3, Idealization and No Conspiracy, the only difference being the number of particles involved. We will refer to them by the same names, with an asterisk added. It is the new quantum mechanical requirement that deserves more attention.

QM Stipulation$*$

QM imposes some constraints on which observation events there are, and accordingly, which intersections of outcomes of events of direction setting are nonempty. The constraints relevant to our purpose are below:[8]

- $z_1^i \cap x_2^j \cap x_3^k \cap x_4^l \neq \emptyset$ iff $i \times j \times k \times l = +1$;
- $z_1^i \cap x_2^j \cap y_3^k \cap y_4^l \neq \emptyset$ iff $i \times j \times k \times l = -1$;
- A set of this kind that contains no z and an even number of xs is non-empty iff it has an odd number of minuses, i.e., $i \times j \times k \times l = -1$.

In what follows, we will be exploring five events:

$$E_{x,x,x,x} \quad E_{y,x,y,x} \quad E_{y,x,x,y} \quad E_{z,x,y,y} \quad E_{z,x,x,x} \tag{*}$$

THEOREM 2
There is no single event CC that is a common-cause for every missing outcome of each event from (*) above.

PROOF (SKETCH). Our proof bears a close similarity to that of Theorem 1. By causal precedence* and QM Stipulation*, there is an atomic outcome o_{CC} that is consistent with an outcome of $E_{x,x,x,x}$. By Idealization* and QM Stipulation*, there must be history h that intersects o_{CC} and contains event $\langle x_1^i, x_2^j, x_3^k, x_4^l \rangle$, where $i \times j \times k \times l = -1$. By Setup Stipulation 2, h must go through each outcome x_1^i, x_2^j, x_3^k and x_4^l and hence

- o_{CC} is consistent with each x_1^i, x_2^j, x_3^k and x_4^l

We take next $y_1^j \cap x_2^j \cap y_3^l \cap x_4^l$, which is missing by QM Stipulation*. By Screening-off and the previously established consistencies,

- o_{CC} is inconsistent with at least one of y_1^j, y_3^l.

No Conspiracy* and Setup Stipulation 3* ensure that

- o_{CC} must be consistent with at least one of y_1^{-j}, y_3^{-l}.

Taking another missing set, namely $y_1^{-j} \cap x_2^j \cap y_3^{-l} \cap x_4^l$, and repeating the argument above we obtain that

- o_{CC} is inconsistent with at least one of y_1^{-j}, y_3^{-l}.

However, both y_1^j and y_3^l cannot be inconsistent with o_{CC}, because then both y_1^{-j} and y_3^{-l} will be consistent with o_{CC}, which contradicts the result above. Similarly, both y_1^{-j} and y_3^{-l} cannot be inconsistent with o_{CC}, either. We thus need to consider two cases, first with o_{CC} inconsistent with y_1^j and consistent with y_3^l and second with o_{CC} inconsistent with y_3^l and consistent with y_1^j. We will go through only the first case here.
Suppose, thus, that y_1^j is inconsistent with o_{CC}, and hence y_1^{-j}—consistent with o_{CC}. This entails that y_3^{-l} is inconsistent with o_{CC}, and hence y_3^l is consistent with o_{CC}.
Let us now take the missing set $y_1^{-j} \cap x_2^j \cap x_3^k \cap y_4^{-k}$. In the light of the consistencies established above, o_{CC} is inconsistent with y_4^{-k}, and hence

- o_{CC} is consistent with y_4^k.

Taking yet another missing $z_1^{-i} \cap x_2^j \cap y_3^l \cap y_4^k$ and repeating the argument above, we first get that o_{CC} is inconsistent with z_1^{-i}, from which it follows that

- o_{CC} is consistent with z_1^i.

Finally, taking the missing set $z_1^i \cap x_2^j \cap x_3^k \cap x_4^l$, we obtain:

- o_{CC} is inconsistent with z_1^i,

which clearly contradicts the result above.
The second case, with o_{CC} being inconsistent with y_3^l and consistent with y_1^j, proceeds similarly and also yields a contradiction.

□

7 Is there a common cause for the EPR (perfect) correlations?

Consider now two stations 1 and 2, the setting of direction at each station being z_i, where z_1 and z_2 are parallel and $z_i^{\pm}$ is an outcome of z_i. As before, we have space-like set: $\{z_1, z_2\}$, and requirements strictly analogical to those of Setup Stipulations and Idealization. (Note that No Conspiracy is trivially satisfied in this setup). The new QM constraint captures the EPR perfect correlations:

QM Stipulation $**$
$z_1^i \cap z_2^j \neq \emptyset$ iff $i \times j = -1$

Now, there is a common cause CC that screens off the missing sets above and that can be produced by the following construction (see Figure 4):

Firstly, to satisfy causal precedence, we locate CC so that it precede each outcome $z_i^{\pm}$. Secondly, in order for conservation to hold, no point x should exist such that $CC \prec x$ and x is incomparable with $z_i^{\pm}$. Given these constraints, there are exactly two atomic outcomes of CC: one consistent with z_1^+, z_2^- and containing $\langle z_1^+, z_2^- \rangle$ (circles in Figure 4) and the other consistent with z_1^-, z_2^+ and containing $\langle z_1^-, z_2^+ \rangle$ (squares in Figure 4). It is not hard to see that screening-off holds: for missing set $z_1^i \cap z_2^j$, $i \times j = +1$, every atomic outcome of CC is inconsistent with either z_1^i or z_2^j. Accordingly, we may explain strict correlations causally, which is, after all, not a big surprise. There is a local hidden-variable model for two spin $1/2$ particles, if the measurement of spin projection is set on parallel directions.[9]

8 Is there a common cause for the GHZ-Bell argument with three particles if directions are fixed?

Let us return to the setup of the Mermin *Gedanken* experiment, but now assume that directions at the three stations are fixed, say, at x_1, x_2 and x_3. According to

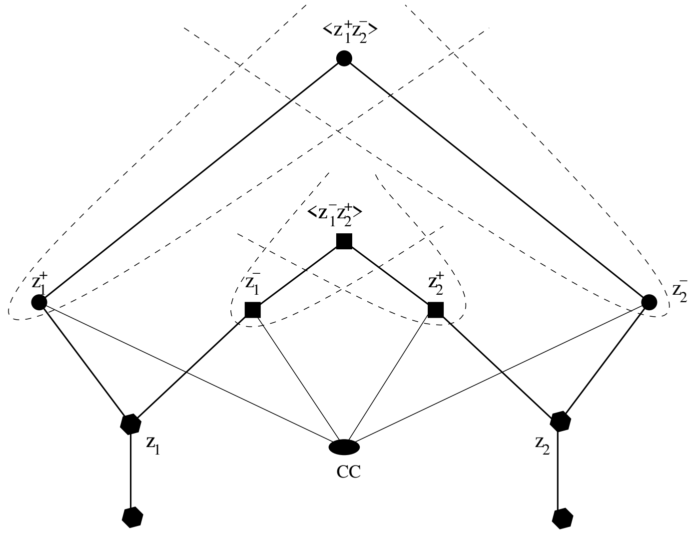

Figure 4: Common cause CC for the EPR correlations. Solid lines represent relation $\prec$ of precedence, the areas within broken lines—outcomes.

QM Stipulation, some outcomes of $E_{x,x,x} = x_1 \cup x_2 \cup x_3$ are missing and these correspond to the missing sets below:

$$x_1^+ \cap x_2^+ \cap x_3^+ \quad x_1^+ \cap x_2^- \cap x_3^- \quad x_1^- \cap x_2^- \cap x_3^+ \quad x_1^- \cap x_2^+ \cap x_3^-$$

As before, we ask whether there is a common-cause that screens off these missing sets? Somewhat surprisingly, yes. To see this, use a similar construction to that suggested by Figure 4, so that CC will have four atomic outcomes that, respectively, are
consistent with each x_1^-, x_2^- and x_3^-;
consistent with each x_1^-, x_2^+ and x_3^+;
consistent with each x_1^+, x_2^- and x_3^+;
consistent with each x_1^+, x_2^+ and x_3^-.

On a closer reflection, the existence of a common cause in the setup considered is nevertheless not so stunning. A typical proof of a Bell-type theorem appeals rather to several sets of directions than a single set. These sets are needed to argue that if at a distant station a direction different from the actual one were chosen, this should not lead to a result distinct from the actual one at the nearby station. Without this sort of appeal to locality, the Bell-type theorems cannot be proved. Another conclusion is that for a proof of the non-existence of common causes, it is essential to have a fuller set of possibilities, i.e., various options of choosing direction settings. You may become immune to the proof by simply refusing to accept that there are any open possibilities in the choice of directions, which is, in essence, denying the independence of the experimenter's choice of the settings from the physical state he or she is in.

An intriguing question, however, is why there are common causes in the last two setups considered? In both the cases we look for a common cause that should screen off missing outcomes of a *single* event. And, it is not hard to see that there is always a common cause for missing outcomes of a single event. Since the formal argument is rather tortuous, here is a simple paper and pencil construction. Suppose $\{E_1, E_2, \ldots, E_n\}$ is a space-like set, so that $\bigcup_i E_i$ is an event and we search for a common cause of missing set $e^* = \bigcap_i e_i$, where e_i is an outcome of E_i for $i = 1, \ldots, n$. Firstly, to satisfy causal precedence, we locate CC so that any two atomic outcomes e_i and e_j and CC form an upward fork (i.e., $\neg\exists_{x \in W}\ CC \prec x \ \&\ x \prec e_i \ \&\ x \prec e_j$ for any $i \neq j$.) This instruction takes care of causal precedence. Secondly, we make sure that there is no point in W that is not in an outcome of some E_i and is preceded by CC. This and the former instruction are to satisfy conservation. It remains to be shown that CC so constructed satisfies screening-off. Observe that in order for an atomic outcome of CC to be consistent with every e_i, where $\bigcap_i e_i = \emptyset$, histories that contain e_i and histories that contain e_j, $i \neq j$, cannot split in CC. However, the way we located CC ensures that these two bundles of histories do split in CC. The argument can be easily extended so as to apply to cases with any larger number of missing sets of the form $\bigcap_i e_i$.

Once there are several events $F^1 = \bigcup_i E_i^1, \ldots, F^n = \bigcup_j E_j^n$, each formed from the corresponding space-like set, and whose missing outcomes a common cause should account for, satisfaction of conservation ceases to be a trivial matter. Thus, the moral is that common causation has a bite only when applied to outcomes of more than one event, that is, if one searches for a common cause that should account for missing outcomes of more than one event.[10]

9 Conclusions

In the first part of the present paper we developed the framework of outcomes in branching space time, the principal result being that the set of outcomes of a (fixed) subset of W is a complete and atomic boolean algebra. We also have a converse result, namely, that every complete boolean algebra is isomorphic to the lattice of outcomes of some event in a certain world W. These are not unwelcome, since, given that outcomes of an event are interpreted as possible results of a measurement procedure, we would prefer that a language describing these results respect classical logic. It may also be considered advantageous that the results could be assigned classical (Kolmogorovean) probability. The algebraic facts do not carry over to the set of outcomes of *more than one* event, however. This set is not, in general, a boolean algebra and it is not even a lattice.

The remaining part of the paper shows how to introduce the concept of non-stochastic common cause to the framework of BST, and subsequently proves some theorems about (non)existence of common causes in Bell-type setups. What appears to us rather surprising is that these theorems, which assert that in some setups common causes are possible and in other setups they are not, mirror the results concerning possibility (impossibility) of contextual local hidden variable models for Bell-type phenomena. A modest conclusion that one can draw from this coincidence says that our definition of the common cause is not completely arbitrary. But those who have no quarrel with this definition being intuitively correct, will no doubt be after a bigger game: it appears that the reason why in some Bell-type setups there are common causes, whereas in other setups there are not, has to do with algebraic structure of outcomes in BST. We already showed that it is always possible to construct a common cause for outcomes of a *single* event. However, it remains to be found what the algebraic necessary and sufficient conditions for the existence of a common cause are.

Let us finally say a few words about prospects for further developments ensuing from this paper. Since the set of outcomes of the event forms a boolean algebra, we may assign classical (Komogorovean) probabilities to it and analyze *stochastic* common causation in quantum setups. Technically, we begin with constructing, for each event of choosing the direction settings, a probability space for the set of outcomes of that event. Next, for each probability space of this kind we appropriately extend it to obtain a probability space on the sets of outcomes of an event consisting of the direction settings *and* a common cause. Finally, we replace our screening-off condition by Reichenbach's common cause principle and modify our requirement of no conspiracy so as it apply to stochastic setups. Then, it may be proved that there is no common cause that accounts for probability distribution of the results of Aspect-Bell experiment (Aspect, Grangier and Roger [1981] and [1982]).

Another issue that comes to mind is a relation between quantum logic and the framework of BST as applied to quantum setups. BST implies that if a measurement can be performed, the outcomes of it form a boolean algebra. The set of outcomes of more than one event, however, is not even a lattice. Nevertheless, it may be hoped that by gluing together some elements of the component boolean algebras, the resulting structure will turn out to be a partial boolean algebra. Given that there is a systematic way of doing this (a procedure known as "pasting" among algebraists), it will provide a link between BST and quantum logic.

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Notes

[1]This view has been advocated by A. Shimony, see for instance his ([1989]).

[2]Compare (Berkovitz [1995]).

[3]This seems to be J. Butterfield's position in ([1992])—compare his PPSI condition.

[4]For more details we refer the reader to (Belnap [1992]).

[5]See (Hofer-Szabó, Rédei and Szabó).

[6]To see how this constraint arises, consult for instance (Mermin [1990b]).

[7]For a more detailed description of the setup, see (Greenberger, Horne, and Zeilinger [1989]) or (Greenberger, Horne, Shimony, and Zeilinger [1990]).

[8]The constraints arise from the quantum mechanical expectation value of the product P of the four results, which is

$$Exp(P) = -\cos(\alpha + \beta - \gamma - \delta),$$

where α, β, γ and δ are the angles at the stations 1, 2, 3 and 4, respectively. To obtain the constraints above, set angles: $\angle x_i = 0$, $\angle y_i = \pi/2$ and $\angle z_1 = \pi$. For details, consult (Greenberger, Horne, Shimony, and Zeilinger [1990]).

[9]See (Bell [1964]).

[10]This is in tally with (Hofer-Szabó, Rédei and Szabó) that answers the question of whether it is possible to extend an original set of events so that an added event serving the role of a common cause satisfies Reichenbach's common cause principle. They show that for a *single* correlation this can always be done. For more than one pair of correlated events, however, it may happen that the set of events does not allow for such an extension.