Quantum mechanics as a representation of classical conditional probabilities

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Questions:

1. How to reconstruct quantum mechanics from classical conditional probabilities?

2. How quantum mechanics is situated within the non-commutative representations of conditional probabilities?

3. Can the specific quantum mechanical representation within this broader frame be traced back to empirical facts or is partly conventional?
Project

Strategy:

• Constructing a simple toy-model
• Representing the model
  • quantum mechanically
  • in a general noncommutative way
• Adding extra empirical facts to the model such that the representation conforms to quantum mechanics
Consider a box filled with balls. Denote the preparation of the box by $s$.

Three measurements:

- $a$: Color measurement
- $b$: Size measurement
- $c$: Shape measurement

Two possible outcomes:

- $A^+$: Black
- $A^-$: White
- $B^+$: Large
- $B^-$: Small
- $C^+$: Round
- $C^-$: Oval
A simple toy-model

- Conditional probabilities:

\[ p_a^\pm := p_s(A^\pm|a) \]
\[ p_b^\pm := p_s(B^\pm|b) \]
\[ p_c^\pm := p_s(C^\pm|c) \]

- Task: represent the above empirical facts by linear operators.

- Question: does the representation conform to quantum mechanics?
Quantum mechanical representation

- **Representation:**

  - **System:** $\mathcal{H}$: Hilbert space
  - **Measurement:** $a \rightarrow O_a$: self-adjoint operator
  - **Outcomes:** $A^i \rightarrow P^i_a$: projections
  - **State:** $s \rightarrow W_s$: density operator

- **Born rule:**

  $$p_s(A^i|a) = \text{Tr}(W_s P^i_a)$$
Quantum mechanical representation

Representation of yes-no measurements:

<table>
<thead>
<tr>
<th>System</th>
<th>$\mathbb{C}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement: $a$</td>
<td>$O_a = a\sigma$, $a \in \mathbb{R}^3$, $</td>
</tr>
<tr>
<td>Outcomes: $A^\pm$</td>
<td>$P_a^\pm = \frac{1}{2}(1 \pm a\sigma)$</td>
</tr>
<tr>
<td>State: $s$</td>
<td>$W_s = \frac{1}{2}(1 + s\sigma)$, $s \in \mathbb{R}^3$, $</td>
</tr>
</tbody>
</table>

$|s| = 1$: pure, $|s| < 1$: mixed state

Born rule:

$$p_s(A^\pm|a) = \text{Tr}(W_sP_a^\pm)$$
Quantum mechanical representation

Representation of the toy-model:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>System</strong></td>
<td>$\mathbb{C}_2$</td>
</tr>
<tr>
<td><strong>Color:</strong> $a$</td>
<td>$O_a = a\sigma$</td>
</tr>
<tr>
<td><strong>Size:</strong> $b$</td>
<td>$O_b = b\sigma$</td>
</tr>
<tr>
<td><strong>Shape:</strong> $c$</td>
<td>$O_c = c\sigma$</td>
</tr>
<tr>
<td><strong>Black/White:</strong> $A^\pm$</td>
<td>$P_a^\pm = \frac{1}{2}(1 \pm a\sigma)$</td>
</tr>
<tr>
<td><strong>Large/Small:</strong> $B^\pm$</td>
<td>$P_b^\pm = \frac{1}{2}(1 \pm b\sigma)$</td>
</tr>
<tr>
<td><strong>Round/Cubic:</strong> $C^\pm$</td>
<td>$P_c^\pm = \frac{1}{2}(1 \pm c\sigma)$</td>
</tr>
<tr>
<td><strong>State:</strong> $s$</td>
<td>$W_s = \frac{1}{2}(1 + s\sigma)$</td>
</tr>
</tbody>
</table>
Quantum mechanical representation

- Born rule:

\[ p_{a}^{\pm} = \frac{1}{2}(1 \pm sa) \]

\[ p_{b}^{\pm} = \frac{1}{2}(1 \pm sb) \]

\[ p_{c}^{\pm} = \frac{1}{2}(1 \pm sc) \]
Quantum mechanical representation
General noncommutative representation
Representation of the toy-model:

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Mapping</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
<td>→ ( \mathbb{C}_2 )</td>
<td></td>
</tr>
<tr>
<td>Color: ( a )</td>
<td>→ ( O_a = a\sigma )</td>
<td></td>
</tr>
<tr>
<td>Size: ( b )</td>
<td>→ ( O_b = b\sigma )</td>
<td></td>
</tr>
<tr>
<td>Shape: ( c )</td>
<td>→ ( O_c = c\sigma )</td>
<td></td>
</tr>
<tr>
<td>Black/White: ( A^\pm )</td>
<td>→ ( P_a^\pm = \frac{1}{2}(1 \pm a\sigma) )</td>
<td></td>
</tr>
<tr>
<td>Large/Small: ( B^\pm )</td>
<td>→ ( P_b^\pm = \frac{1}{2}(1 \pm b\sigma) )</td>
<td></td>
</tr>
<tr>
<td>Round/Cubic: ( C^\pm )</td>
<td>→ ( P_c^\pm = \frac{1}{2}(1 \pm c\sigma) )</td>
<td></td>
</tr>
<tr>
<td>State: ( s )</td>
<td>→ ( W_s = ? )</td>
<td></td>
</tr>
</tbody>
</table>
General noncommutative representation

Measurement-operator assignment:

\[ \{ a, b, c \} \mapsto \{ a, b, c \} \]

Proposition:

\[ p_a^\pm, p_b^\pm, p_c^\pm \quad \& \quad a, b, c \implies W_s \]

Specifically:

\[ s = \frac{(p_a^+ - \frac{1}{2})(b \times c) + (p_b^+ - \frac{1}{2})(c \times a) + (p_c^+ - \frac{1}{2})(a \times b)}{a \cdot (b \times c)} \]
Question: How to fix

\[ \{a, b, c\} \mapsto \{a, b, c\} \]

such that \( W_s \) becomes a density operator (that is \(|s| \leq 1\))?  

Strategy: Enrich the model by further details

1. Performing a fourth measurement
2. Performing joint measurements
3. Measuring two systems
4. Symmetry considerations?
5. Adopting the projection postulate
6. Assuming elements of reality
7. Adopting the projection postulate and assuming elements of reality
How to assign operators to measurements?

1. Performing a fourth measurement
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- $d$: weight measurement
- $p_d^\pm$: conditional probabilities
How to assign operators to measurements?

1. Performing a fourth measurement

- $d$: weight measurement
- $p_d^\pm$: conditional probabilities
2. Performing joint measurements
2. Performing joint measurements

- Commensurability:

\[ p_{ab}^{\pm\pm} = p_s(A^\pm \land B^\pm | a \land b) \]

- Representation of four outcomes:

<table>
<thead>
<tr>
<th>System:</th>
<th>( \mathbb{C}_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement:</td>
<td>( O_{ab} )</td>
</tr>
<tr>
<td>Outcomes:</td>
<td>( P_{ab}^{\pm\pm} )</td>
</tr>
<tr>
<td>State:</td>
<td>( W'_s )</td>
</tr>
</tbody>
</table>

- Born rule:

\[ p_{ab}^{\pm\pm} = \text{Tr}(W'_s P_{ab}^{\pm\pm}) \]
3. Measuring two systems
How to assign operators to measurements?

3. Measuring two systems

\[ p_a^{\pm}, p_b^{\pm}, p_c^{\pm} \ & a, b, c \implies W_s \]
3. Measuring two systems

\[ p_a^\pm, p_b^\pm, p_c^\pm \quad \& \quad a, b, c \implies W_s \]

\[ p_a'^\pm, p_b'^\pm, p_c'^\pm \quad \& \quad a, b, c \implies W_{s'} \]
3. Measuring two systems

\[ p_a^\pm, p_b^\pm, p_c^\pm \quad \& \quad a, b, c \implies W_s \]

\[ p_a'^\pm, p_b'^\pm, p_c'^\pm \quad \& \quad a, b, c \implies W_{s'} \]

\[ q p_a^\pm + (1-q)p_a'^\pm \]

\[ q p_b^\pm + (1-q)p_b'^\pm \quad \& \quad a, b, c \]

\[ q p_c^\pm + (1-q)p_c'^\pm \]
3. Measuring two systems

\[ p_a^\pm, p_b^\pm, p_c^\pm \ & \ a, b, c \implies W_s \]

\[ p_a'^\pm, p_b'^\pm, p_c'^\pm \ & \ a, b, c \implies W_s' \]

\[ q p_a^\pm + (1 - q)p_a'^\pm \]

\[ q p_b^\pm + (1 - q)p_b'^\pm \ & \ a, b, c \]

\[ q p_c^\pm + (1 - q)p_c'^\pm \]

\[ \implies W_{qs+(1-q)s'} \]
4. Symmetry considerations?
How to assign operators to measurements?

4. Symmetry considerations?

- Representation of the spin

  Spin measurement in direction $a \rightarrow O_a = a\sigma$

  Outcomes: $A^\pm \rightarrow P_a^\pm = \frac{1}{2}(1 \pm a\sigma)$

- Representation of a rotation group
How to assign operators to measurements?

4. Symmetry considerations?
5. Adopting the projection postulate
5. Adopting the projection postulate

How many of the balls first measured to be black will turn out to be large for a second, size measurement?
5. Adopting the projection postulate

- Pre-measurement state: $W_s$
How to assign operators to measurements?

5. Adopting the projection postulate

- Pre-measurement state: $W_s$
- Post-measurement state: $P_a^+$
How to assign operators to measurements?

5. Adopting the projection postulate

- Pre-measurement state: $W_s$
- Post-measurement state: $P_a^+$
- Probability of outcome $B^+$:

$$p_{b|a}^+ = \text{Tr}(P_a^+ P_b^+) = \frac{1}{2} (1 + ab)$$
5. Adopting the projection postulate

- Pre-measurement state: $W_s$
- Post-measurement state: $P_a$
- Probability of outcome $B^+$:

$$p_{b|a^+} = \text{Tr}(P_a^+ P_b^+) = \frac{1}{2} (1 + ab)$$

- Suppose that $p_{b|a^\pm}$, $p_{b|c^\pm}$, $p_{c|b^\pm}$ are empirically given
5. Adopting the projection postulate

\[ p_{b|a}^{±} = \frac{1}{2}(1 + (±1 \cdot ±1)ab) \]
\[ p_{b|c}^{±} = \frac{1}{2}(1 + (±1 \cdot ±1)ac) \]
\[ p_{c|b}^{±} = \frac{1}{2}(1 + (±1 \cdot ±1)bc) \]
6. Assuming elements of reality
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Element of reality:

\[ p_s(A^\pm | a \land \alpha^\pm) = \delta_{\pm\pm} \]
\[ p_s(a \land \alpha^\pm) = p_s(a) p_s(\alpha^\pm) \]

Consequence:

\[ p_s(A^\pm | a) = p_s(\alpha^\pm) \]
6. Assuming elements of reality

\[ p_s(\alpha^{\pm}) = \frac{1}{2}(1 \pm sa) \]
7. Adopting the projection postulate and assuming elements of reality
How to assign operators to measurements?

7. Adopting the projection postulate and assuming elements of reality

- Requirement:

\[ p_a^± = p_s(α^±) = \frac{1}{2}(1 ± sa) \]

\[ p_{a|b}^± = p_s(α^±|β^±) = \frac{1}{2}(1 + (±1 · ±1)ab) \]

- Can be satisfied if

\[ s ⊥ a, b, c \quad \text{or} \quad a ∥ b ∥ c \]

- But there is no restriction to the length of \( s \)!
Conclusions

**Question:** Is quantum mechanics the only possible way to represent an empirically given set of classical conditional probabilities in a noncommutative way?

**Strategy:**
- Constructing a simple toy-model
- Representing the model in a general noncommutative way
- Adding extra empirical facts to the model such that the representation conforms to quantum mechanics

**Answer:** Empirical facts do not restrict the representation to be quantum mechanical. The quantum mechanical representation is, at least partly, conventional.


