Two concepts of noncontextuality in quantum mechanics

Gábor Hofer-Szabó

Research Center for the Humanities, Budapest
Main messages

- There are two different and logically independent concepts of noncontextuality in quantum mechanics: simultaneous and measurement noncontextuality.

- There are no state-independent Kochen-Specker arguments ruling out simultaneous noncontextual hidden variable models for QM.
A hidden variable model for QM is noncontextual if every hidden state determines the probability distribution of outcomes of every measurement independently of what other measurements are simultaneously performed.
A hidden variable model for QM is noncontextual if

- every hidden state determines the probability distribution of outcomes of every measurement independently of what other measurements are simultaneously performed

- any two measurements which are represented by the same self-adjoint operator have the same probability distribution of outcomes in every hidden state
A hidden variable model for QM is noncontextual if

- every hidden state determines the probability distribution of outcomes of every measurement independently of what other measurements are simultaneously performed (simultaneous noncontextuality)

- any two measurements which are represented by the same self-adjoint operator have the same probability distribution of outcomes in every hidden state (measurement noncontextuality)
Simultaneous contextuality:
  - Bell, Shimony
  - Heywood and Redhead’s environmental contextuality

Measurement contextuality:
  - Kochen and Specker
  - Van Fraassen’s ontological contextuality
  - Spekkens’ measurement contextuality
  - Bohmian mechanics
KS theorems
<table>
<thead>
<tr>
<th>$\sigma_z \otimes 1$</th>
<th>$1 \otimes \sigma_z$</th>
<th>$\sigma_z \otimes \sigma_z$</th>
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<tbody>
<tr>
<td>$1 \otimes \sigma_x$</td>
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<td>$\sigma_y \otimes \sigma_y$</td>
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GHZ pentagram

\[ \sigma_y \otimes 1 \otimes 1 \]

\[ \sigma_x \otimes \sigma_x \otimes \sigma_x \quad \sigma_y \otimes \sigma_y \otimes \sigma_x \quad \sigma_y \otimes \sigma_x \otimes \sigma_y \quad \sigma_x \otimes \sigma_y \otimes \sigma_y \]

\[ 1 \otimes 1 \otimes \sigma_x \quad 1 \otimes 1 \otimes \sigma_y \]

\[ \sigma_x \otimes 1 \otimes 1 \]

\[ 1 \otimes \sigma_y \otimes 1 \quad 1 \otimes \sigma_x \otimes 1 \]
KS theorems
KS theorems

- Value assignments:

\[ v : \{O_i\} \rightarrow R; \quad v(O_i) \in \sigma(O_i) \]

- **Functional composition principle:** for any set \( \{O_1, O_2, \ldots \} \) of pairwise commuting operators

\[
\text{if } f(O_1, O_2, \ldots) = 0, \text{ then } f(v(O_1), v(O_2), \ldots) = 0
\]
KS theorems

\[
\begin{align*}
\sigma_z \otimes 1 & \quad 1 \otimes \sigma_z & \quad \sigma_z \otimes \sigma_z \\
1 \otimes \sigma_x & \quad \sigma_x \otimes 1 & \quad \sigma_x \otimes \sigma_x \\
\sigma_z \otimes \sigma_x & \quad \sigma_x \otimes \sigma_z & \quad \sigma_y \otimes \sigma_y
\end{align*}
\]
KS theorems

Value assignments
Functional composition principle
\[ \implies \text{Contradiction} \]
KS arguments
KS arguments

KS theorems  +  physical interpretation
EPR-Bell arguments
EPR-Bell arguments
KS arguments
KS arguments
Commutativity $\iff$ Simultaneous measurability
KS arguments
If \( \{A_i\} \) are pairwise commuting operators, then there is an operator \( B \) and functions \( \{f_i\} \) such that \( A_i = f_i(B) \).
KS arguments

Diagram:

- Left: Black grid
- Right: Red and blue grid
KS arguments
KS arguments
KS arguments
Conclusion

There are no state-independent Kochen-Specker arguments ruling out simultaneous noncontextual hidden variable models for QM
More on that . . .

- Gábor Hofer-Szabó, “Commutativity, comeasurability, and contextuality in the Kochen-Specker arguments” (submitted)
- Gábor Hofer-Szabó, “Two concepts of noncontextuality in quantum mechanics” (submitted)
GHZ argument
Operational theory

- $S$: preparations
- $M$: measurements
- $X_a$: outcomes of measurement $a \in M$

Probability distributions:

$$\{p(A_i|a \land s) : A_i \in X_a, a \in M, s \in S\}$$
Quantum mechanics

- Preparations $\rightarrow$ density operators
- Measurements $\rightarrow$ self-adjoint operators
- Outcomes $\rightarrow$ eigenvalues (spectral projections)

Born rule:

$$\text{Tr}(\rho P^a_i) = p(A_i|a \land s) \quad A_i \in X_a, \ a \in M, \ s \in S$$
Hidden variable model

Probability distributions:

\[ \{ p(\lambda|s) : \lambda \in \Lambda, \ s \in S \} \]

Response functions:

\[ \{ p(A_i|a \land \lambda) : A_i \in X_a, \ a \in M, \ \lambda \in \Lambda \} \]

such that

\[ p(A_i|a \land s) = \sum_{\lambda \in \Lambda} p(A_i|a \land \lambda) p(\lambda|s) \quad A_i \in X_a, \ a \in M, \ s \in S \]
Two concepts of noncontextuality

Simultaneous noncontextuality:

\[ p(A_i | a \land \lambda) = p(A_i | a \land b \land \lambda) \quad A_i \in X_a, \ a, b, a \land b \in M, \ \lambda \in \Lambda \]

Measurement noncontextuality: If

\[ p(A_i | a \land s) = p(B_j | b \land s) \quad A_i = B_j, \ A_i \in X_a, \ B_j \in X_b, \ s \in S \]

then

\[ p(A_i | a \land \lambda) = p(B_j | b \land \lambda) \quad A_i = B_j, \ A_i \in X_a, \ B_j \in X_b, \ \lambda \in \Lambda \]
“The danger in fact was not in the explicit but in the implicit assumptions. It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously.”
“[W]e have seen that these operators commute and it is a generally accepted assumption of quantum mechanics that commuting operators correspond to commensurable observables. A rationale for this assumption . . . is that if $A_i, i \in I$ is a set of mutually pairwise commuting self-adjoint operators, then there exists a self-adjoint operator $B$ and Borel functions $f_i, i \in I$ such that $A_i = f_i(B)$. However, this justification hinges on the existence of a physical observable which corresponds to the operator.”

Physical observable = spin-Hamiltonian: “the change in the energy of the lowest orbital state of orthohelium resulting from the application of a small electric field with rhombic symmetry”
\[
[\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y] = \sigma_x \otimes \sigma_x \cdot \sigma_y \otimes \sigma_y - \sigma_y \otimes \sigma_y \cdot \sigma_x \otimes \sigma_x \\
= \sigma_x \sigma_y \otimes \sigma_x \sigma_y - \sigma_y \sigma_x \otimes \sigma_y \sigma_x \\
= i\sigma_z \otimes i\sigma_z - (-i)\sigma_z \otimes (-i)\sigma_z \\
= -\sigma_z \otimes \sigma_z + \sigma_z \otimes \sigma_z \\
= 0
\]