Three levels of Bell’s inequalities

Gábor Hofer-Szabó
Research Centre for the Humanities, Budapest
I. The Pitowsky representation – Bell’s inequalities for classical probabilities

II. The Common Cause Principle – Bell’s inequalities for classical conditional probabilities

III. Bell’s inequalities for quantum probabilities
I. The Pitowsky representation
Question: When can numbers represent probabilities?

- Consider real numbers $p_i$ and $p_{ij}$ in $[0, 1]$ such that $i = 1 \ldots n$ and $(i, j) \in S$ where $S$ is a subset of the index pairs $\{(i, j) | i < j; i, j = 1 \ldots n\}$

- When can these numbers be probabilities of events $A_i$ and their conjunctions $A_i \land A_j$?
I. The Pitowsky representation

Pitowsky’s geometrical answer:

- Correlation vector: \( p = (p_1, p_2, \ldots, p_n; \ldots, p_{ij}, \ldots) \)

- Real \( n + |S| \) dimensional vector space: \( R(n, S) \cong \mathbb{R}^{n+|S|} \)
  (where \( |S| \) is the cardinality of \( S \))

- Let \( \varepsilon \in \{0, 1\}^n \). To each \( \varepsilon \) assign a vertex vector (truth-value functions): \( u^\varepsilon \in R(n, S) \) such that
  \[
  u^\varepsilon_i = \varepsilon_i \quad i = 1 \ldots n \\
  u^\varepsilon_{ij} = \varepsilon_i \varepsilon_j \quad (i, j) \in S
  \]
I. The Pitowsky representation

Classical correlation polytopes:

\[ c(n, S) := \left\{ v \in R(n, S) \mid v = \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon u^\varepsilon ; \lambda_\varepsilon \geq 0; \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon = 1 \right\} \]
I. The Pitowsky representation

Remarks:

- Pitowsky theorem, 1989: $p$ has a Kolmogorovian representation iff $p \in c(n, S')$

- The classical correlation polytopes can be characterized by their facet inequalities; these are Bell’s inequalities.
Example 1: $n = 2$, $S = \{(1, 2)\}$:

- Vertices: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 1)$

- Correlation polytope:

- Facet inequalities:

\[
0 \leq p_{12} \leq p_1 \leq 1 \\
0 \leq p_{12} \leq p_2 \leq 1 \\
p_1 + p_2 - p_{12} \leq 1
\]
I. The Pitowsky representation

Example 2: \( n = 3, \ S = \{(1, 2), (1, 3), (2, 3)\} \):

- Facet inequalities (Bell-Pitowsky inequalities):

\[
\begin{align*}
0 \leq p_{ij} &\leq p_i \leq 1 \\
0 \leq p_{ij} &\leq p_j \leq 1 \\
p_i + p_j - p_{ij} &\leq 1 \\
p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} &\leq 1 \\
p_1 - p_{12} - p_{13} + p_{23} &\geq 0 \\
p_2 - p_{12} - p_{23} + p_{13} &\geq 0 \\
p_3 - p_{13} - p_{23} + p_{12} &\geq 0
\end{align*}
\]
I. The Pitowsky representation

Example 3: \( n = 4, S = \{(1, 3), (1, 4), (2, 3), (2, 4)\} \):

- Facet inequalities (Clauser-Horne-Pitowsky inequalities):

\[
0 \leq p_{ij} \leq p_i \leq 1 \\
0 \leq p_{ij} \leq p_j \leq 1 \quad i = 1, 2 \quad j = 3, 4 \\
p_i + p_j - p_{ij} \leq 1 \\
-1 \leq p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leq 0 \\
-1 \leq p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leq 0 \\
-1 \leq p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leq 0 \\
-1 \leq p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leq 0
\]
Remarks:

For \( n > 4 \) facet inequalities are not known. The complexity of derivation of the facet inequalities grows exponentially with \( n \).

Quantum probabilities cannot represent probability of properties or events since their correlation vector is not in the classical correlation polytope.
II. The Common Cause Principle
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The Common Cause Principle:

If there is a correlation between two events and there is no direct causal connection between the correlating events, then there always exists a common cause of the correlation.
II. The Common Cause Principle

The common cause:

- Classical probability space: \((\Sigma, p)\)
- Correlation: \(A, B \in \Sigma\)

\[
p(AB) \neq p(A)p(B)
\]

- Common cause: partition \(\{C_k\}_{k \in K} \) in \(\Sigma\)

\[
p(AB|C_k) = p(A|C_k)p(B|C_k)
\]

- Sometimes \(\Sigma\) needs to be extended in order to contain a common cause.
II. The Common Cause Principle

The EPR-Bohm scenario:

- Measurements: $a_i, b_j$ ($i, j = 1, 2$: different directions)
- Outcomes: $A_i, B_j$ ($i, j = 1, 2$: up, down)
II. The Common Cause Principle

The EPR-Bohm scenario:

- Conditional probabilities:

\[
p(A_i | a_i) = \frac{1}{2} \\
p(B_j | b_j) = \frac{1}{2} \\
p(A_i \land B_j | a_i \land b_j) = \frac{1}{2} \sin^2 \left( \frac{\theta_{ij}}{2} \right)
\]

where \( \theta_{ij} \) is the angle between directions \( a_i \) and \( b_j \).

- Conditional correlations for non-perpendicular directions:

\[
p(A_i \land B_j | a_i \land b_j) \neq p(A_i | a_i)p(B_j | b_j)
\]
What is the common cause of the correlation?
II. The Common Cause Principle

Common causal explanation:

\[ p(A_i \land B_j | a_i \land b_j \land C_k) = p(A_i | a_i \land b_j \land C_k) \cdot p(B_j | a_i \land b_j \land C_k) \]
\[ p(A_i | a_i \land b_j \land C_k) = p(A_i | a_i \land C_k) \]
\[ p(B_j | a_i \land b_j \land C_k) = p(B_j | b_j \land C_k) \]
\[ p(a_i \land b_j \land C_k) = p(a_i \land b_j) \cdot p(C_k) \]
II. The Common Cause Principle

However, a common causal explanation implies the Clauser-Horne inequality:

\[-1 \leq p(A_1 \land B_3 | a_1 \land b_3) + p(A_1 \land B_4 | a_1 \land b_4) + p(A_2 \land B_3 | a_2 \land b_3) - p(A_2 \land B_4 | a_2 \land b_4) - p(A_1 | a_1) - p(B_3 | b_3) \leq 0\]

...which are violated for certain measurement angles:

\[\theta_{13} = \theta_{14} = \theta_{23} = 120^\circ \text{ and } \theta_{24} = 0^\circ\]

Hence, EPR correlations do not have a (local, non-conspiratorial) common causal explanation.
III. Quantum Bell inequalities
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Local physical theory:

- Net: association of von Neumann algebras to spacetime regions
- Events, $A_i, B_j$: projections localized in spacetime regions
- State, $\phi$: a normalized positive linear functional
III. Quantum Bell inequalities

Quantum Bell inequality:

- Clauser–Horne–Summers inequality:

\[-1 \leq \phi(A_1B_3 + A_1B_4 + A_2B_3 - A_2B_4 - A_1 - B_3) \leq 0\]

- Quantum common cause: a set of mutually orthogonal projections \(\{C_k\}\) such that

\[
\frac{\phi(C_kA_iB_jC_k)}{\phi(C_k)} = \frac{\phi(C_kA_iC_k)}{\phi(C_k)} \frac{\phi(C_kB_jC_k)}{\phi(C_k)}
\]

- However, the quantum Bell inequalities follow from the quantum common cause only if it is commuting
Bell’s inequality à la carte:

I. Clauser-Horne-Pitowsky inequality:

\[-1 \leq p(A_1 \land B_3) + p(A_1 \land B_4) + p(A_2 \land B_3) - p(A_2 \land B_4) - p(A_1) - p(B_3) \leq 0\]

II. Clauser-Horne inequality:

\[-1 \leq p(A_1 \land B_3|a_1 \land b_3) + p(A_1 \land B_4|a_1 \land b_4) + p(A_2 \land B_3|a_2 \land b_3) - p(A_2 \land B_4|a_2 \land b_4) - p(A_1|a_1) - p(B_3|b_3) \leq 0\]

III. Clauser–Horne–Summers inequality:

\[-1 \leq \phi(A_1 B_3 + A_1 B_4 + A_2 B_3 - A_2 B_4 - A_1 - B_3) \leq 0\]
Conclusion

To what question is Bell’s inequality an answer?

I. Question: When can numbers represent probabilities?
II. Question: When do correlations have a common causal explanation?
III. Question: ???