COMMON CAUSAL EXPLANATIONS AND BELL'S INEQUALITIES

Gábor Hofer-Szabó

Eötvös University Budapest

Email: gsz@szig.hu

Question

How are the following two facts related?

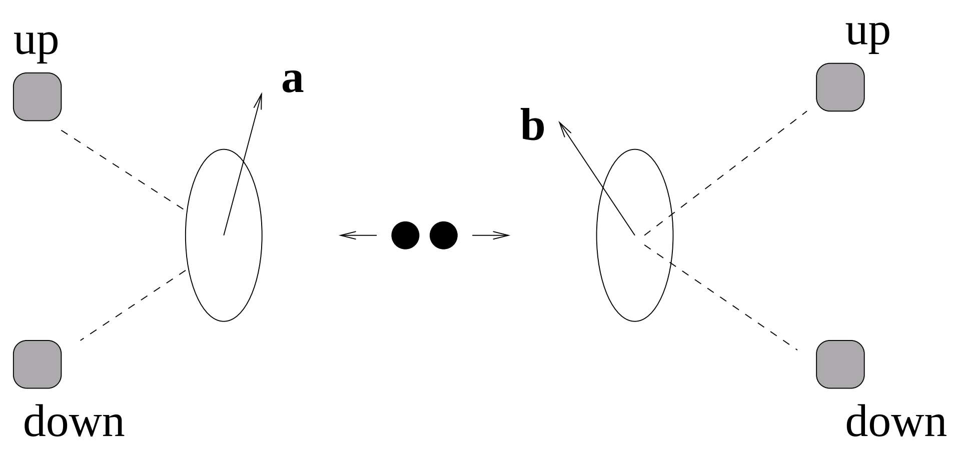
(i) A set of correlations has a local, non-conspiratorial *separate* common causal explanation;

(ii) the set satisfies Bell's inequalities.

Project

- The EPR-Bohm scenario
- What is a common cause, a common cause system, a *common* common cause system, etc.?
- *Common vs. separate* common causal explanation of the EPR scenario
- Separate common causal explanation and Bell's inequalities

EPR experiment



Measurement settings:

Left wing: a_i Right wing: b_j $(i, j \in J)$

Measurement outcomes:

Left wing: A_i, $\overline{A}_i$ Right wing: B_j, $\overline{B}_j$

Conditional probabilities

Conditional probabilities:

$$p(A_iB_j|a_ib_j) = Tr(W_{|\Psi_s\rangle}(P_{A_i} \otimes P_{B_j})) = \frac{1}{2} sin^2(\frac{\theta_{a_ib_j}}{2})$$

$$p(A_i|a_ib_j) = Tr(W_{|\Psi_s\rangle}(P_{A_i} \otimes I)) = \frac{1}{2}$$

$$p(B_j|a_ib_j) = Tr(W_{|\Psi_s\rangle}(I \otimes P_{B_j})) = \frac{1}{2}$$

- Tr: trace function
- $W_{|\Psi_s\rangle}$: density operator pertaining to the pure state $|\Psi_s\rangle$
- P_{A_i} and P_{B_j}: projections on the eigensubspaces with eigenvalue $+1$ of the spin operators associated with directions $\vec{a}_i$ and $\vec{b}_j$
- $\theta_{a_ib_j}$: angle between directions $\vec{a}_i$ and $\vec{b}_j$

Conditional correlations

- **Conditional correlations:** for a_i, b_j non-orthogonal directions

$$p(A_i B_j | a_i b_j) \neq p(A_i | a_i b_j) p(B_j | a_i b_j)$$

Conditional correlations

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$$p(A_i B_j | a_i b_j) \neq p(A_i | a_i b_j) p(B_j | a_i b_j)$$

- **Perfect anticorrelation:** for a_i, b_j parallel directions

$$p(A_i B_j | a_i b_j) = 0$$

The problem

- **Classical probability measure space:** (Ω, Σ, p)

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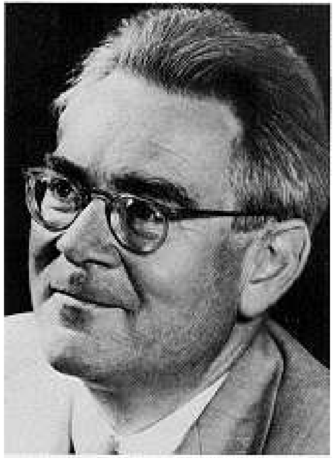
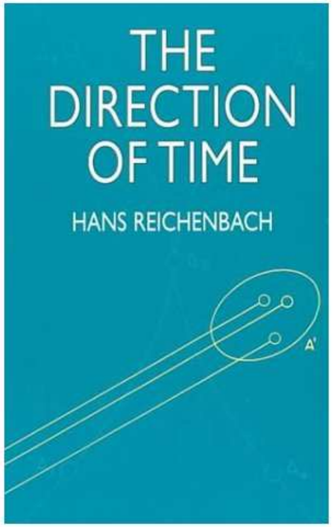
The problem

- **Classical probability measure space:** (Ω, Σ, p)
- **Set of (conditionally) correlating pairs:** $\{(A_i, B_j)\}_{i,j \in I}$
- **Question:** Is there a common causal explanation of the set $\{(A_i, B_j)\}_{i,j \in I}$?
- **What is**
  - a common cause;
  - a common cause system;
  - a *common* common cause system;
  - a set of separate common cause systems?

Reichenbach: The Direction of Time





The Origin of the Common Cause?

Russell: common causal ancester: ”When a group of complex events in more or less the same neighbourhood and ranged about a central event all have a common structure, it is probable that they have a common causal ancester.” (*Human Knowledge*, p. 483)

- ”A number of middle-aged ladies in different parts of the country, after marrying and insuring their lives in favour of their husbands, mysteriously died in the baths. The identity of structure between these different events led to the assumption of a common causal origin; this origin was found to be Mr. Smith, who was duly hanged.” (p. 482)

Reichenbachian common cause

- **Classical probability measure space:** (Ω, Σ, p)

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- **Positive correlation:** A, $B \in \Sigma$

$$p(AB) > p(A)p(B)$$

Reichenbachian common cause

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- **Positive correlation:** A, $B \in \Sigma$

$$p(AB) > p(A)p(B)$$

- **Reichenbachian common cause:** $C \in \Sigma$

$$\begin{aligned} p(AB|C) &= p(A|C)p(B|C) \\ p(AB|\overline{C}) &= p(A|\overline{C})p(B|\overline{C}) \\ p(A|C) &> p(A|\overline{C}) \\ p(B|C) &> p(B|\overline{C}) \end{aligned}$$

Common cause system

- **Correlation:** $A, B \in \Sigma$

$$p(AB) \neq p(A)p(B)$$

- **Common cause system:** n-partition $\{C_k\}_{k \in K}$ of Σ

$$p(AB|C_k) = p(A|C_k)p(B|C_k)$$

Common vs. separate common cause system

- **Set of correlating pairs:** $\{(A_i, B_j)\}_{i,j \in I}$

$$p(A_i B_j) \neq p(A_i)p(B_j)$$

Common vs. separate common cause system

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$$p(A_i B_j) \neq p(A_i) p(B_j)$$

- ***Common* common cause system:** a partition $\{C_k\}_{k \in K}$ of Σ

$$p(A_i B_j | C_k) = p(A_i | C_k) p(B_j | C_k)$$

Common vs. separate common cause system

- **Set of correlating pairs:** $\{(A_i, B_j)\}_{i,j\in I}$

$$p(A_iB_j) \neq p(A_i)p(B_j)$$

- ***Common* common cause system:** a partition $\{C_k\}_{k\in K}$ of Σ

$$p(A_iB_j|C_k) = p(A_i|C_k)p(B_j|C_k)$$

- **Separate common cause systems:** a set of partitions $\left\{C_k^{ij}\right\}_{k(i,j)\in K(i,j)}$ of Σ

$$p(A_iB_j|C_k^{ij}) = p(A_i|C_k^{ij})p(B_j|C_k^{ij})$$

EPR: *Common* common causal explanation

- **A *common* common causal explanation** of the set $\{(A_i, B_j)\}_{i,j \in I}$ $(A_i, B_j, a_i, b_j \in \Sigma)$ consists in providing a partition $\{C_k\}_{k \in K}$ of Σ such that the following three requirements hold:

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- **Screening-off:**

$$p(A_i B_j | a_i b_j C_k) = p(A_i | a_i b_j C_k) p(B_j | a_i b_j C_k)$$

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$$p(A_i B_j | a_i b_j C_k) = p(A_i | a_i b_j C_k) p(B_j | a_i b_j C_k)$$

- **Locality:**

$$p(A_i | a_i b_j C_k) = p(A_i | a_i C_k) \;\; p(B_j | a_i b_j C_k) = p(B_j | b_j C_k)$$

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- **Locality:**

$$p(A_i | a_i b_j C_k) = p(A_i | a_i C_k) \;\; p(B_j | a_i b_j C_k) = p(B_j | b_j C_k)$$

- **No-conspiracy:** for any element of $\{C_k\}_{k\in K}$

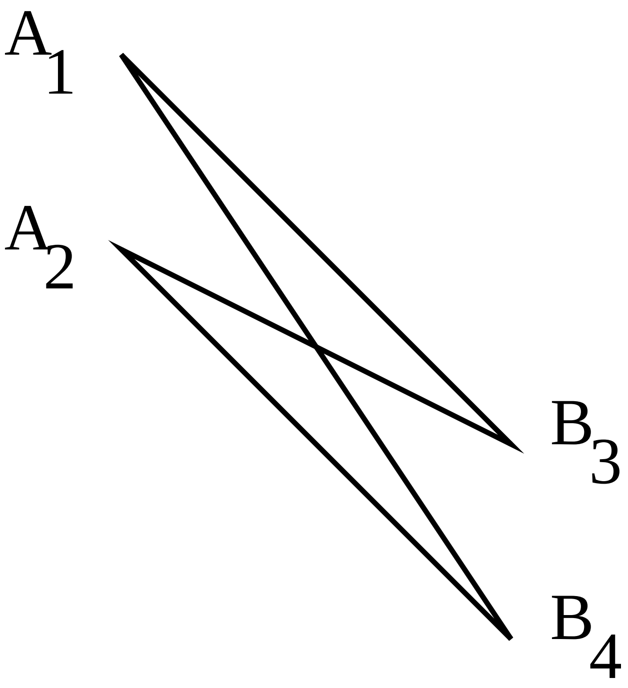
$$p(a_i b_j C_k) = p(a_i b_j) p(C_k)$$

Bell's inequalities

Common common cause system
Locality $\Longrightarrow$ Bell's inequalities
No-conspiracy

Clauser–Horne correlation set:

$$\{(A_i, B_j)\}_{CH} \equiv \{(A_1, B_3), (A_1, B_4), (A_2, B_3), (A_2, B_4)\}$$



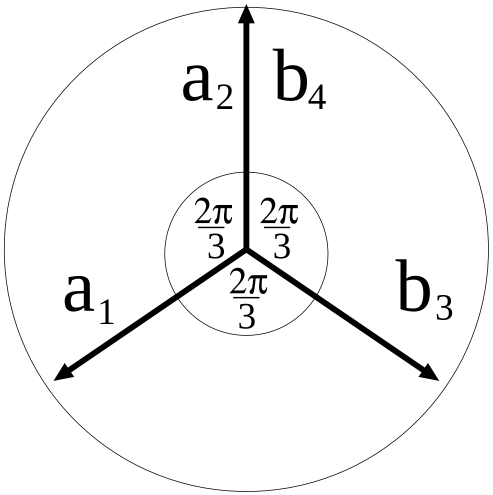
Clauser–Horne inequality

Proposition: Let $\{(A_i, B_j)\}_{CH}$ be the Clauser–Horne correlation set where A_i, B_j, a_i and b_j $(i = 1, 2; j = 3, 4)$ are elements of a classical probability measure space (X, S, p). Suppose that $\{(A_i, B_j)\}_{CH}$ has a local, non-conspiratorial *common* common causal explanation in the above sense. Then for any $i, i' = 1, 2; j, j' = 3, 4$; $i \neq i', j \neq j'$ the following Clauser–Horne inequality follows:

$$-1 \leqslant p(A_i B_j | a_i b_j) + p(A_i B_{j'} | a_i b_{j'}) + p(A_{i'} B_j | a_{i'} b_j)$$
$$-p(A_{i'} B_{j'} | a_{i'} b_{j'}) - p(A_i | a_i b_j) - p(B_j | a_i b_j) \leqslant 0$$

Violation of the Clauser–Horne inequality

For the following setting:



the Clauser–Horne inequality is violated at the upper bound:

$$\frac{3}{8}+\frac{3}{8}+\frac{3}{8}-0-\frac{1}{2}-\frac{1}{2} \;\not\leq\; 0$$

Consequently, $\{(A_i, B_j)\}_{CH}$ can not be given a local, non-conspiratorial *common* common causal explanation.

Bell's inequalities

- *Common* common cause system
  Locality $\Longrightarrow$ Bell inequality
  No-conspiracy

Bell's inequalities

- *Common* common cause system
  Locality $\Longrightarrow$ Bell inequality
  No-conspiracy
- However, having a *common* common cause system is a *very strong* requirement!

L. E. Szabó 's question

- **Question:** "Whether there exist separate common causes for the correlations observed in the EPR-Aspect experiment"? (Szabó, 2000)

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- *Separate* common cause systems
  Locality $\stackrel{?}{=}$ EPR
  No-conspiracy

EPR: Separate common causal explanation

- **A separate common causal explanation** of the set $\{(A_i, B_j)\}_{i,j\in I}$ $(A_i, B_j, a_i, b_j \in \Sigma)$ consists in providing a separate partition $\left\{C_k^{ij}\right\}_{k(i,j)\in K(i,j)}$ of Σ for each correlation of $\{(A_i, B_j)\}_{i,j\in I}$ such that the following requirements hold:

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-

$$p(A_iB_j|a_ib_jC_k^{ij}) = p(A_i|a_ib_jC_k^{ij})p(B_j|a_ib_jC_k^{ij}) \qquad \text{(screening-off)}$$
$$p(A_i|a_ib_jC_k^{ij}) = p(A_i|a_iC_k^{ij}) \qquad \text{(locality)}$$
$$p(B_j|a_ib_jC_k^{ij}) = p(B_j|b_jC_k^{ij}) \qquad \text{(locality)}$$
$$p(a_ib_jF) = p(a_ib_j)p(F) \qquad \text{(no-conspiracy)}$$

No-conspiracy

- **No-conspiracy:** $F \in \mathfrak{C} \subset \Omega$ generated by the C_{ij}-s

$$p(a_i b_j F) = p(a_i b_j) p(F)$$

No-conspiracy

- **No-conspiracy:** $F \in \mathfrak{C} \subset \Omega$ generated by the C_{ij}-s

$$p(a_i b_j F) = p(a_i b_j) p(F)$$

- **'Reduced' no-conspiracy:**

$$p(a_i b_j C_{kl}) = p(a_i b_j) p(C_{kl})$$

Szabó 's conjecture

- ". . . combinations of the common cause events as $C_{kl}C_{km}$, $C_{kl} \cup C_{km}$, $C_{kl}C_{km}C_{nl}$ etc. do statistically correlate with the measurement operations."

$$p(a_i b_j C_{kl} C_{mn}) \neq p(a_i)p(b_j)p(C_{kl}C_{mn})$$

Szabó 's conjecture

- ”...combinations of the common cause events as $C_{kl}C_{km}$, $C_{kl} \cup C_{km}$, $C_{kl}C_{km}C_{nl}$ etc. do statistically correlate with the measurement operations.”

$$p(a_i b_j C_{kl} C_{mn}) \neq p(a_i)p(b_j)p(C_{kl}C_{mn})$$

- **Conjecture:** There exists no local, non-conspiratorial separate-common-cause-model for the EPR.

The Bern group project

- ”Whether a model can be constructed without these correlations [conspiracies] was posed as an open question by Szabó. This question is answered negatively by the derivation of Bell’s inequality.” (Graßhoff, Portmann, Wüthrich, 2005, p. 668.)

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- *Separate* common cause systems
  Locality $\Longrightarrow$ Bell inequality
  No-conspiracy

A shortcoming: perfect anticorrelations

”We have not been able to derive a Bell-type inequality ruling out perfect correlations and allowing different common cause variables [separate common causes]. If PCORR [perfect correlation] is indeed a necessary assumption for our derivation of the Bell inequality, it should be possible to construct a model in which PCORR [perfect correlation] does not hold (being violated by arbitrary small deviation, say). Since the actually measured correlations are never perfect—a fact that is usually attributed to experimental imperfections—it is not obvious how such a model could be refuted.” (Graßhoff et al., 2005, p. 677.)

Unwelcome corollaries

In case of perfect anticorrelations:

- The verification of Szabó's conjecture is sensitive to experimental imperfections.
- In case of perfect anticorrelations the set of separate common cause systems can be reduced to a *common* common cause system. (Hofer-Szabó, 2008)

Improvements

Improvements: *Almost* perfect anticorrelations

- No separate common causal explanation for *almost* perfect EPR anticorrelations (Portmann, Wüthrich, 2007; Hofer-Szabó, 2008)
- Bell(δ) inequalities from separate common causal explanation for *almost* perfect EPR anticorrelations (Hofer-Szabó, 2011a)

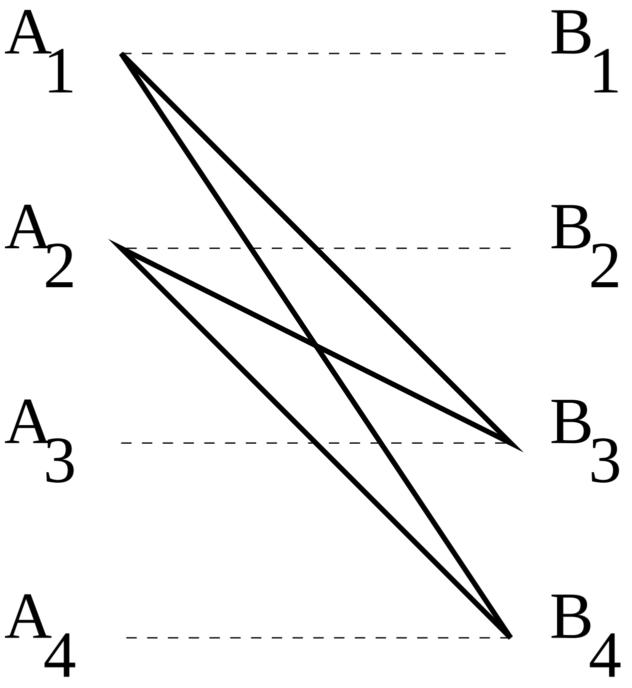
Chronology

- Belnap, Szabó, 1996: *Common* common causes and separate common causes are different.
- Szabó, 2000: Is there a *separate* common causal explanation of the EPR scenario?
- Graßhoff, Portmann, Wüthrich, 2005: No separate common causal explanation for perfect EPR anticorrelations.
- Portmann, Wüthrich, 2007; Hofer-Szabó, 2008: No separate common causal explanation for *almost* perfect EPR anticorrelations.
- Hofer-Szabó, 2011a: Bell(δ) inequalities from separate common causal explanation for *almost* perfect EPR anticorrelations.

However . . .

There is something very embarrassing in the proofs!

- Consider the following two sets of correlations:



Two sets of correlations

Clauser–Horne correlation set:
$\{(A_i, B_j)\}_{CH} \equiv \{(A_i, B_j)\}_{i=1,2;\, j=3,4}$ such that $\theta_{a_i b_j}$ between the directions $\vec{a}_i$ and $\vec{b}_j$ are set as follows:

$$\theta_{a_1 b_3} = \theta_{a_1 b_4} = \theta_{a_2 b_3} = \frac{2\pi}{3} \quad \text{and} \quad \theta_{a_2 b_4} = 0$$

Perfect anticorrelation set: $\{(A_i, B_i)\}_{PA} \equiv \{(A_i, B_i)\}_{i=1,2,3,4}$ such that for any $i = 1, 2, 3, 4$ the angle $\theta_{a_i b_i} = 0$

A question and an answer

- **Szabó's question:** Is there a local, non-conspiratorial separate common causal model for the set $\{(A_i, B_j)\}_{CH}$?

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- **Szabó's question:** Is there a local, non-conspiratorial separate common causal model for the set $\{(A_i, B_j)\}_{CH}$?
- **Answer:** A necessary condition for $\{(A_i, B_j)\}_{PA}$ to have a local, non-conspiratorial separate common causal explanation is that $\{(A_i, B_j)\}_{CH}$ satisfies the Clauser–Horne inequality

$$-1 \leqslant p(A_i B_j | a_i b_j) + p(A_i B_{j'} | a_i b_{j'}) + p(A_{i'} B_j | a_{i'} b_j) \\ -p(A_{i'} B_{j'} | a_{i'} b_{j'}) - p(A_i | a_i b_j) - p(B_j | a_i b_j) \leqslant 0$$

Sketch of the proof I.

Suppose that $\{(A_i, B_i)\}_{PA}$ has a local, non-conspiratorial separate common causal explanation: $\{C_k^{ii}\}_{k\in K(i)}$. Since $\{(A_i, B_i)\}_{PA}$ consists of only perfect anticorrelations it is easy to show that for any $i = 1, 2, 3, 4$ there exist a vector $\varepsilon^{ii} \in \{0, 1\}^{K(i)}$ such that defining C^{ii} and $C^{ii\perp}$ as

$$C^{ii} \equiv \bigcup_{k\in K(i)} \varepsilon_k^{ii}\, C_k^{ii}; \qquad C^{ii\perp} \equiv \bigcup_{k\in K(i)} (1 - \varepsilon_k^{ii})\, C_k^{ii}$$

the partitions $\{C^{ii}, C^{ii\perp}\}$ $(i = 1, 2, 3, 4)$ will be deterministic local, non-conspiratorial separate common *causes* and

$$\begin{aligned} p(C^{ii}) &= p(A_i|a_i b_i) \\ p(C^{ii\perp}) &= p(B_i|a_i b_i) \end{aligned}$$

Sketch of the proof II.

Using locality and no-conspiracy *for* $\{(A_i, B_j)\}_{PA}$! one obtains that for any $i, j = 1, 2, 3, 4$; $i \neq j$

$$p(C^{ii}) = p(A_i|a_i b_j) \quad (1)$$

$$p(C^{jj\perp}) = p(B_j|a_i b_j) \quad (2)$$

$$p(C^{ii}C^{jj\perp}) = p(A_i B_j|a_i b_j) \quad (3)$$

For the four events C^{ii}, $C^{i'i'}$, $C^{jj\perp}$ and $C^{j'j'\perp}$ it holds that:

$$-1 \leqslant p(C^{ii}C^{jj\perp}) + p(C^{ii}C^{j'j'\perp}) + p(C^{i'i'}C^{33\perp}) - p(C^{i'i'}C^{j'j'\perp}) - p(C^{ii}) - p(C^{jj\perp}) \leqslant 0 \quad (4)$$

Plugging (1)-(3) into (4) we get the Clauser–Horne inequality *for* $\{(A_i, B_j)\}_{CH}$!

Upshot

- To put is briefly, the necessary condition for $\{(A_i, B_j)\}_{PA}$ to have a local, non-conspiratorial separate common causal explanation is that $\{(A_i, B_j)\}_{CH}$ satisfies the Clauser–Horne inequality!

Upshot

- To put is briefly, the necessary condition for $\{(A_i, B_j)\}_{PA}$ to have a local, non-conspiratorial separate common causal explanation is that $\{(A_i, B_j)\}_{CH}$ satisfies the Clauser–Horne inequality!
- The papers (Portmann and Wüthrich, 2007) and (Hofer-Szabó, 2008, 2011a) have repeated the same argumentation for *almost* perfect anticorrelations. In this case we arrive at some *Bell(δ) inequalities* differing from the original Bell inequlities in a term of order of δ.

What has and what has not been proven

- This answer is perfectly adequate if our intention is to exclude the local, non-conspiratorial separate common causal explanation of the EPR scenario *as such*—as was the aim of the paper (Graßhoff et al. 2005).
- But it *does not at all explain* the fact why Szabó was not able to give a local, non-conspiratorial separate common causal explanation of his original set $\{(A_i, B_j)\}_{CH}$.

Repeating the question

Question: Is there a local, non-conspiratorial separate common causal model for the set $\{(A_i, B_j)\}_{CH}$?

The answer

- **Not known.**

The answer

- **Not known.**
- **A partial answer:** Let $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ be a set of correlating pairs such that A_i, B_j, a_i and b_j are elements of a classical probability measure space (X, S, p). Suppose furthermore that $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ has a local, non-conspiratorial separate common causal explanation which is *deterministic* in the sense that for any $i = 1, 2$; $j = 3, 4$ and $k(ij) \in K(i, j)$

$$p(A_i|a_i b_j C_k^{ij}),\ \ p(B_j|a_i b_j C_k^{ij}) \in \{0, 1\}$$

Then for any $i, i' = 1, 2; j, j' = 3, 4$; $i \neq i', j \neq j'$ the Clauser–Horne inequality follows. (Hofer-Szabó, 2011b)

Conclusion

- The question as to why Szabó was unable to provide a local, non-conspiratorial separate common causal explanation for the $\{(A_i, B_j)\}_{CH}$ set is still open.

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Proof of the Clauser–Horne inequality

Proof. Trivial arithmetical fact: for any $\alpha, \alpha', \beta, \beta' \in [0, 1]$

$$-1 \leqslant \alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' - \alpha - \beta \leqslant 0 \tag{5}$$

Let $\alpha, \alpha', \beta, \beta'$ be the following conditional probabilities:

$$\alpha \equiv p(A_i|a_i b_j C_k) \tag{6}$$

$$\alpha' \equiv p(A_{i'}|a_{i'} b_{j'} C_k) \tag{7}$$

$$\beta \equiv p(B_j|a_i b_j C_k) \tag{8}$$

$$\beta' \equiv p(B_{j'}|a_{i'} b_{j'} C_k) \tag{9}$$

Plugging (6)-(9) into (5), using locality and screening-off one obtains

$$-1 \leqslant p(A_i B_j|a_i b_j C_k) + p(A_i B_{j'}|a_{i'} b_j C_k) + p(A_{i'} B_j|a_{i'} b_j C_k)$$
$$-p(A_{i'} B_{j'}|a_{i'} b_{j'} C_k) - p(A_i|a_i b_j C_k) - p(B_j|a_i b_j C_k) \leqslant 0$$

Multiplying by $p(C_k)$, summing up for the indices k and using no-conspiracy one obtains the above Clauser–Horne inequalities. □