Bell inequality and common causal explanation in algebraic quantum field theory

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Question: What is the relation between
  - the Bell inequalities
  - and the common causal explanation of correlations
in algebraic quantum field theory (AQFT)?
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Quantum case: Bell inequality
**Question:** What is the relation between the Bell inequalities and the common causal explanation of correlations in algebraic quantum field theory (AQFT)?

**Classical case:** Common cause $\implies$ Bell inequality

**Quantum case:** ? $\implies$ Bell inequality
I. Classical common causal explanation
II. Nonclassical common causal explanation
III. One correlation: Common Cause Principles in AQFT
IV. More correlations: joint common causal explanation in AQFT
Reichenbachian common cause
"Suppose several actors in a stage play fall ill showing symptoms of food poisoning. We assume that the poisoned food stems from the same source – for instance, that it was contained in a common meal – and then look for an explanation of the coincidence in terms of a common cause."
Reichenbachian common cause

- **Classical probability space:** \((\Sigma, p)\)

- **Positive correlation:** \(A, B \in \Sigma\)

\[
p(AB) > p(A)p(B)
\]

- **Reichenbachian common cause:** \(C \in \Sigma\)

\[
\begin{align*}
p(AB|C) &= p(A|C)p(B|C) \\
p(AB|\overline{C}) &= p(A|\overline{C})p(B|\overline{C}) \\
p(A|C) &> p(A|\overline{C}) \\
p(B|C) &> p(B|\overline{C})
\end{align*}
\]
Common cause system

- **Correlation:** \( A, B \in \Sigma \)

\[
p(AB) \neq p(A)p(B)
\]

- **Common cause system (CCS):** partition \( \{C_k\}_{k \in K} \) in \( \Sigma \)

\[
p(AB|C_k) = p(A|C_k)p(B|C_k)
\]

- **Common cause:** CCS of size 2.
Localization of the common cause
Localization of the common cause

Weak common cause system
Localization of the common cause

Common cause system
Localization of the common cause

Strong common cause system
Localization of the common cause

$V_A$ and $V_B$: localization of $A$ and $B$.

Weak past: $wpast(V_A, V_B) := I_-(V_A) \cup I_-(V_B)$

Common past: $cpast(V_A, V_B) := I_-(V_A) \cap I_-(V_B)$

Strong past: $spast(V_A, V_B) := \cap_{x \in V_A \cup V_B} I_-(x)$
Joint common cause system

**Correlations:** \( A_m, B_n \in \Sigma \ (m \in M, n \in N) \)

\[ p(A_m B_n) \neq p(A_m) p(B_n) \]

**Joint CCS:** partition \( \{C_k\}_{k \in K} \) in \( \Sigma \)

\[ p(A_m B_n | C_k) = p(A_m | C_k) p(B_n | C_k) \]
Conditional probabilities

- **Measurement outcomes:** $A_m, B_n \in \Sigma (m \in M, n \in N)$
- **Measurement choices:** $a_m, b_n \in \Sigma (m \in M, n \in N)$ also localized in $V_A$ and $V_B$
- **Conditional correlations:**

$$p(A_m B_n | a_m b_n) \neq p(A_m | a_m) p(B_n | b_n)$$
Local, non-conspiratorial joint common causal explanation: a partition \( \{C_k\} \) in \( \Sigma \)

\[
p(A_mB_n|a_mb_nC_k) = p(A_m|a_mb_nC_k)p(B_n|a_mb_nC_k) \quad \text{(screening-off)}
\]

\[
p(A_m|a_mb_nC_k) = p(A_m|a_mC_k) \quad \text{(locality)}
\]

\[
p(B_n|a_mb_nC_k) = p(B_n|b_nC_k) \quad \text{(locality)}
\]

\[
p(a_mb_nC_k) = p(a_mb_n)p(C_k) \quad \text{(no-conspiracy)}
\]

\( \{C_k\} \) can be localized in any of the three different pasts
Motivation by Markov condition

Markov condition $\iff$ screening-off, locality and no-conspiracy
Clauser–Horne inequality

Joint CCS
Locality $\implies$ CH inequality
No-conspiracy

\[-1 \leq p(A_1 B_1 | a_1 b_1) + p(A_1 B_2 | a_1 b_2) + p(A_2 B_1 | a_2 b_1)\]
\[-p(A_2 B_2 | a_2 b_2) - p(A_1 | a_1) - p(B_1 | b_1) \leq 0\]

(which is equivalent to the CHSH inequality used in AQFT.)
EPR correlations

- Conditional probabilities:

\[ p(A_m|a_m), \; p(B_n|b_n), \; p(A_mB_n|a_mb_n) \quad (m, n = 1, 2) \]

- CH inequality is violated.

- Therefore: no common causal explanation for EPR.
”Bell inequalities are relations between conditional probabilities valid under the locality assumption.” (Gisin, 2009)
Quantum ontology

- **Event space**: von Neumann lattice
- **Events**: projections
- **Probability**: quantum state
- **CH inequality**:

\[-1 \leq \phi (A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 - A_1 - B_1) \leq 0\]
Quantum ontology

- **Event space**: von Neumann lattice
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Then let us take the quantum ontology seriously!

- **Common cause**:
  1. What is that?
  2. How it relates to the CH inequality?
Non-classical common cause system

- **Non-classical probability space:** \((\mathcal{P}(\mathcal{N}), \phi)\)

- **Correlation:** \(A, B \in \mathcal{P}(\mathcal{N})\)

\[\phi(AB) \neq \phi(A)\phi(B)\]

- **(Non-classical) CCS:** partition \(\{C_k\}_{k \in K}\) in \(\mathcal{P}(\mathcal{N})\)

\[
\frac{\phi(C_k ABC_k)}{\phi(C_k)} = \frac{\phi(C_k AC_k)}{\phi(C_k)} \frac{\phi(C_k BC_k)}{\phi(C_k)}
\]
Non-classical common cause system

- **Non-classical probability space:** \((\mathcal{P}(\mathcal{N}), \phi)\)
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  \[\phi(AB) \neq \phi(A)\phi(B)\]
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  \[
  \frac{\phi(C_k ABC_k)}{\phi(C_k)} = \frac{\phi(C_k AC_k)}{\phi(C_k)} \frac{\phi(C_k BC_k)}{\phi(C_k)}
  \]
- **Commuting / Noncommuting CCS:** \(\{C_k\}_{k \in K}\) is commuting / not commuting with \(A\) and \(B\)
- **Nontrivial CCS:** \(C_k \not\subset A, A^\perp, B\) or \(B^\perp\) for some \(k \in K\)
Joint common cause system

- **Set of correlations:** $A_m, B_n \in \mathcal{P}(\mathcal{N})$

  $$\phi(A_m B_n) \neq \phi(A_m) \phi(B_n)$$

- **Joint CCS:** partition $\{C_k\}_{k \in K}$ in $\mathcal{P}(\mathcal{N})$

  $$\frac{\phi(C_k A_m B_n C_k)}{\phi(C_k)} = \frac{\phi(C_k A_m C_k)}{\phi(C_k)} \frac{\phi(C_k B_n C_k)}{\phi(C_k)}$$

- **A subtle point:**

  Joint CCS = local, non-conspiratorial joint CCS
Clauser–Horne inequality

Commuting joint CCS

(Locality) $\implies$ CH inequality
(No-conspiracy)

$-1 \leq \phi (A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 - A_1 - B_1) \leq 0$

EPR has no commutative common causal explanation
But why to demand commutativity between a cause and its effects?

Standard QM: operators do not commute with their time translates:

Harmonic oscillator: $x(t) \equiv U(t)^{-1}xU(t)$

$$[x(t), x] \psi_0 = -\frac{i\hbar}{m\omega} \sin (\hbar\omega t)\psi_0 \neq 0$$
Noncommuting common causes

*Noncommuting* joint CCS

(Locality) $\not\Rightarrow$ CH inequality

(No-conspiracy)

**Question:** Can a set of correlations violating the CH inequality have a noncommuting *joint* common causal explanation in AQFT?
An easier question: Can one correlation have a common causal explanation in AQFT? (Rédei 1997)

Common Cause Principle (CCP): If there is a correlation between two events and there is no direct causal (or logical) connection between them, then there always exists a common cause of the correlation.
Algebraic quantum field theory

- **Axioms:**
  - (i) Isotony
  - (ii) Einstein causality
  - (iii) Relativistic covariance

- **Representation:**
  - (iv) Irreducible vacuum representation
  - (v) Weak additivity
  - (vi) Type III von Neumann algebras
  - (vii) Local primitive causality
Algebraic quantum field theory

- Poincaré covariant AQFT:

- Quantum Ising model:
Weak (Commutative/Noncommutative) CCP
(Commutative/Noncommutative) CCP
Strong (Commutative/Noncommutative) CCP
**Proposition:** The Weak Commutative CCP holds in Poincaré covariant AQFT (Rédei, Summers, 2002).

**Question:** What about other AQFTs?

Question: What about abandoning commutativity?
**Original question:** Can a set of correlations violating the CH inequality have a noncommuting joint common causal explanation in AQFT?

Quantum Ising model ...
Quantum Ising model

Minimal double cones: $\mathcal{O}_{m_i}^m$
Quantum Ising model

**Double cones:** $\mathcal{O}_{i,j}$, smallest double cone containing $\mathcal{O}_{i}^{m}$ and $\mathcal{O}_{j}^{m}$
Net: $\kappa^m$, by integer time translation
‘One-point’ algebras

- **Linear basis:** $1, U_0$
- **Minimal projections:** $P = \frac{1}{2} (1 \pm U_0)$
- **Commutation relations:**

$$U_i U_j = \begin{cases} 
-U_j U_i, & \text{if } |i - j| = \frac{1}{2} \\
U_j U_i, & \text{otherwise}
\end{cases}$$
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U_j U_i, & \text{otherwise}
\end{cases}$$
‘Two-point’ algebras

- **Linear basis:** $1, U_0, U_{1/2}, iU_0U_{1/2}$
- **Minimal projections:** $P = \frac{1}{2} \left( 1 + \vec{n} \cdot U \right), \quad \vec{n} \in \mathbb{R}^3$
**Dynamics:** automorphisms of $\mathcal{A}$ (Müller, Vecsernyés 2012)

- Local primitive causality holds.
**Dynamics**

- **Dynamics**: automorphisms of $\mathcal{A}$ (Müller, Vecsernyés 2012)
- Local primitive causality holds.
Correlations violating CH

\[ A_m = A(\vec{a}^m), \quad B_n = B(\vec{b}^n): \text{four projections } (m, n = 1, 2) \]

\[ \rho^s: \text{singlet state} \]
Correlations violating CH

Directions:

maximally violating of the CH inequality ...
... or, equivalently, the CHSH inequality:

\[ |\phi(U_1(V_1 + V_2) + U_2(V_1 - V_2))| \leq 2 \]

where

\[ U_m := 2A_m - 1 \]
\[ V_n := 2B_n - 1 \]
Correlations violating CHSH

Question: Can these four correlations have a noncommutative joint common causal explanation?
... after some calculation ...

\[ \rho_C = 1 + \lambda U_{-\frac{1}{2}} U_{\frac{1}{2}} \]

\[ + \frac{1 + \lambda}{2} c_1(U_{-\frac{1}{2}} + U_{\frac{1}{2}}) + \frac{1 - \lambda}{2} c_1'(U_{-\frac{1}{2}} - U_{\frac{1}{2}}) \]

\[ + \frac{1 + \lambda}{2} c_2(U_0 - U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) - \lambda c_2(U_{-1} U_0 U_1 + U_{-1} U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}} U_1) \]

\[ + \frac{1 - \lambda}{2} c_2'(U_0 + U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) \]

\[ + \frac{1 + \lambda}{2} c_3 i(U_{-\frac{1}{2}} U_0 - U_0 U_{\frac{1}{2}}) + \frac{1 - \lambda}{2} c_3' i(U_{-\frac{1}{2}} U_0 + U_0 U_{\frac{1}{2}}) \]

\[ + \lambda c_1 c_2(U_{-1} U_{-\frac{1}{2}} U_0 U_1 + U_{-1} U_0 U_{\frac{1}{2}} U_1) \]

\[ + \lambda c_2^2 (-U_{-1} U_1 + U_{-1} U_{-\frac{1}{2}} U_{\frac{1}{2}} U_1) \]

\[ + \lambda c_2 c_3 i(U_{-1} U_{-\frac{1}{2}} U_1 - U_{-1} U_{\frac{1}{2}} U_1). \]

Answer: Yes.
Localization of the common cause

Weak joint common cause system
Localization of the common cause

Joint common cause system
Localization of the common cause

Strong joint common cause system
Localization of the common cause

Weak joint common cause system: one needs only local primitive causality and isotony (no dynamics)
(Strong) joint common cause system: one needs also dynamics
Noncommuting common causes

**Proposition:** (Hofer-Szabó, Vecsernyés, 2012b, 2013b)
There is a noncommuting common cause \( \{C, C^\perp\} \) of the correlations \( \{(A_m, B_n)\} \); and it can be localized in the shaded region.
Conclusion

**Classical case:**  Common cause $\implies$ Bell inequality

**Quantum case:**  Bell inequality
Classical case: Common cause $\Rightarrow$ Bell inequality

Quantum case: $\nRightarrow$ Bell inequality

The violation of the Bell inequality in AQFT does not exclude a set of correlations to have a joint common causal explanation if commutativity is abandoned.
References


V.F. Müller and P. Vecsernyés, "The phase structure of $G$-spin models", *to be published*.


Remarks

- In the noncommutative case the theorem of total probability does \textit{not} hold. (No 'Hempelian' explanation.)
- Are the (Strong/Weak) Noncommutative \textit{Joint} CCPs valid in AQFT?
- What are the ontological consequences of applying noncommutative common causes?
Bell inequality in AQFT

- \( \mathcal{A} \) and \( \mathcal{B} \): two mutually commuting \( C^* \)-subalgebras of \( C \)
- **Bell operator** for \((\mathcal{A}, \mathcal{B})\): \( R \), an element of the set

\[
\mathbb{B}(\mathcal{A}, \mathcal{B}) \equiv \left\{ \frac{1}{2} (A_1(B_1 + B_2) + A_1(B_1 - B_2)) \right\} \\
A_i = A_i^* \in \mathcal{A}; \ B_i = B_i^* \in \mathcal{B}; \ -1 \leq A_i, B_i \leq 1
\]
Bell inequality in AQFT

- Bell correlation coefficient of a state $\phi$:

$$\beta(\phi, A, B) \equiv \sup \{ |\phi(R)| \mid R \in \mathcal{B}(A, B) \}$$

- The Bell inequality is violated if

$$|\beta(\phi, A, B)| > 1$$
Proposition: If $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras then there are some states violating the Bell inequality for $\mathcal{A} \otimes \mathcal{B}$ iff both $\mathcal{A}$ and $\mathcal{B}$ are non-abelian (Bacciagaluppi, 1994).

- Going over to von Neumann algebras ... (Landau 1987)
- Adding further constraints ... (Summer-Werner, 1988; Halvorson, Clifton, 2000)
- The above theorems apply in "typical" AQFTs ...
Joint common cause system

Joint CCS = local, non-conspiratorial joint CCS

Proof:

- Rewriting both the classical and the non-classical local, non-conspiratorial joint CCS in an *indexical form*.

- ’Translating’ quantum probabilities into classical conditional probabilities by the *Kolmogorovian Censorship Hypothesis*. 
Correlation:

\[ \phi(A_m B_n) \neq \phi(A_m) \phi(B_n) \]

Indexical notation:

\[ \phi_{C_k}(X) := \frac{(\phi \circ E_c)(XC_k)}{\phi(C_k)} = \frac{\phi(C_kXC_k)}{\phi(C_k)}. \]

Non-classical, local, non-conspiratorial joint CCS:

\[
\begin{align*}
\phi_{C_k}(A_m B_n) &= \phi_{C_k}(A_m) \phi_{C_k}(B_n) \\
\phi_{C_k}(A_m) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m B_n^\perp) \\
\phi_{C_k}(B_n) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m B_n^\perp) \\
\phi_{C_k}(1) &= 1.
\end{align*}
\]
Kolmogorovian Censorship Hypothesis

Let \( (\mathcal{N}, \mathcal{P}(\mathcal{N}), \phi) \) be a non-classical probability space. Let \( \Gamma \) be a countable set of non-commuting selfadjoint operators in \( \mathcal{N} \). For every \( Q \in \Gamma \), let \( \mathcal{P}(Q) \) be a maximal Abelian sublattice of \( \mathcal{P}(\mathcal{N}) \) containing all the spectral projections of \( Q \). Finally, let a map \( p_0 : \Gamma \to [0, 1] \) be such that

\[
\sum_{Q \in \Gamma} p_0(Q) = 1, \quad p_0(Q) > 0.
\]

Then there exists a classical probability space \( (\Omega, \Sigma, p) \) such that for every projection \( X^Q \) in any \( \mathcal{P}(Q) \) there exist events \( X^Q_{cl} \) and \( x^Q_{cl} \) in \( \Sigma \) such that

\[
X^Q_{cl} \subset x^Q_{cl},
\]

\[
x^Q_{cl} \cap x^R_{cl} = 0, \quad \text{if} \ Q \neq R
\]

\[
p(x^Q_{cl}) = p_0(Q)
\]

\[
\phi(X^Q) = p(X^Q_{cl} | x^Q_{cl})
\]
Classical joint common cause system

Correlation:
\[ p(A_m \land B_n \mid a_m \land b_n) \neq p(A_m \mid a_m) p(B_n \mid b_n) \]

Indexical notation:
\[ p_{C_k}(X \mid x) := \frac{p(X \land C_k \mid x)}{p(C_k)}. \]

Classical, local, non-conspiratorial joint CCS:
\[ p_{C_k}(A_m \land B_n \mid a_m \land b_n) = p_{C_k}(A_m \mid a_m \land b_n) p_{C_k}(B_n \mid a_m \land b_n), \]
\[ p_{C_k}(A_m \mid a_m \land b_n) = p_{C_k}(A_m \mid a_m \land b_n'), \]
\[ p_{C_k}(B_n \mid a_m \land b_n) = p_{C_k}(B_n \mid a_m' \land b_n), \]
\[ p_{C_k}(\Omega \mid a_m \land b_n) = 1. \]
Quantum Ising model

Cauchy surface net: $K_{CS}^m$, poset of double cones based on the Cauchy surface
‘Three-point’ algebras

- **Linear basis:**
  \[ 1, U_{-\frac{1}{2}}, U_0, U_{\frac{1}{2}}, iU_{-\frac{1}{2}}U_0, iU_0U_{\frac{1}{2}}, U_{-\frac{1}{2}}U_{\frac{1}{2}}, U_{-\frac{1}{2}}U_0U_{\frac{1}{2}} \]

- **Minimal projections:** \( P = P(\overrightarrow{n}), \quad \overrightarrow{n} \in \mathbb{R}^3 \)

- **Two dimensional projections:** \( P = P(\overrightarrow{n}, \overrightarrow{n}'), \quad \overrightarrow{n}, \overrightarrow{n}' \in \mathbb{R}^3 \)
1. **Isotony.** The net is given by the isotone map
\( \mathcal{K} \ni V \mapsto \mathcal{A}(V) \) to unital \( \mathcal{C}^* \)-algebras, that is \( V_1 \subseteq V_2 \) implies that \( \mathcal{A}(V_1) \) is a unital \( \mathcal{C}^* \)-subalgebra of \( \mathcal{A}(V_2) \). The **quasilocal algebra** \( \mathcal{A} \) is the inductive limit \( \mathcal{C}^* \)-algebra of the net \( \{ \mathcal{A}(V), V \in \mathcal{K} \} \) of local \( \mathcal{C}^* \)-algebras.

2. **Microcausality (Einstein causality):**
\( \mathcal{A}(V')' \cap \mathcal{A} \supseteq \mathcal{A}(V), V \in \mathcal{K} \), where primes denote spacelike complement and algebra commutant, respectively.

3. **Spacetime covariance.** A group homomorphism
\( \alpha : \mathcal{P}_\mathcal{K} \to \text{Aut} \mathcal{A} \) is given such that the automorphisms \( \alpha_g, g \in \mathcal{P}_\mathcal{K} \) of \( \mathcal{A} \) act covariantly on the observable net:
\( \alpha_g(\mathcal{A}(V)) = \mathcal{A}(g \cdot V), V \in \mathcal{K}. \)
Non-classical common cause system

- **Conditional expectation:**

\[ E_c : \mathcal{N} \rightarrow \mathcal{C}, \quad A \mapsto \sum_{k \in K} C_k AC_k \]

- **(Non-classical) CCS:** partition \( \{C_k\}_{k \in K} \) in \( \mathcal{P}(\mathcal{N}) \)

\[
\frac{(\phi \circ E_c)(ABC_k)}{\phi(C_k)} = \frac{(\phi \circ E_c)(AC_k)}{\phi(C_k)} \cdot \frac{(\phi \circ E_c)(BC_k)}{\phi(C_k)}
\]

- **Commuting / Noncommuting CCS:** \( \{C_k\}_{k \in K} \) is commuting / not commuting with \( A \) and \( B \)

- **Nontrivial CCS:** \( C_k \not\leq A, A^\perp, B \) or \( B^\perp \) for some \( k \in K \)