Bell inequality and common causal explanation in algebraic quantum field theory

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**Question:** What is the relation between the Bell inequalities and the common causal explanation of correlations in algebraic quantum field theory (AQFT)?
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- the Bell inequalities
- and the common causal explanation of correlations in algebraic quantum field theory (AQFT)?

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**Quantum case:** Bell inequality $\downarrow$
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**Classical case:** Common cause $\implies$ Bell inequality

**Quantum case:** ? $\implies$ Bell inequality
I. Classical common causal explanation
II. Nonclassical common causal explanation
III. One correlation: Common Cause Principles in AQFT
IV. More correlations: joint common causal explanation in AQFT
Reichenbachian common cause

- Classical probability space: \((\Sigma, p)\)
- Positive correlation: \(A, B \in \Sigma\)
  \[
p(AB) > p(A)p(B)
  \]
- Reichenbachian common cause: \(C \in \Sigma\)
  \[
p(AB|C) = p(A|C)p(B|C)
p(AB|\bar{C}) = p(A|\bar{C})p(B|\bar{C})
p(A|C) > p(A|\bar{C})
p(B|C) > p(B|\bar{C})
  \]
Common cause system

- **Correlation:** \( A, B \in \Sigma \)
  \[
p(AB) \neq p(A)p(B)
  \]

- **Common cause system (CCS):** partition \( \{C_k\}_{k \in K} \) in \( \Sigma \)
  \[
p(AB|C_k) = p(A|C_k)p(B|C_k)
  \]

- **Common cause:** CCS of size 2.
Localization of the common cause
Localization of the common cause

Weak common cause system
Localization of the common cause

- Common cause system
Localization of the common cause

- Strong common cause system
Localization of the common cause

\[ V_A \text{ and } V_B: \text{ localization of } A \text{ and } B. \]

Weak past: \[ wpast(V_A, V_B) := I_-(V_A) \cup I_-(V_B) \]
Common past: \[ cpast(V_A, V_B) := I_-(V_A) \cap I_-(V_B) \]
Strong past: \[ spast(V_A, V_B) := \cap_{x \in V_A \cup V_B} I_-(x) \]
Joint common cause system

- **Correlations:** $A_m, B_n \in \Sigma \ (m \in M, n \in N)$

  \[ p(A_m B_n) \neq p(A_m) p(B_n) \]

- **Joint CCS:** partition $\{C_k\}_{k \in K}$ in $\Sigma$

  \[ p(A_m B_n | C_k) = p(A_m | C_k) p(B_n | C_k) \]
Conditional probabilities

Measurement outcomes: $A_m, B_n \in \Sigma \ (m \in M, n \in N)$

Measurement choices: $a_m, b_n \in \Sigma \ (m \in M, n \in N)$ also localized in $V_A$ and $V_B$

Conditional correlations:

$$p(A_m B_n | a_m b_n) \neq p(A_m | a_m) p(B_n | b_n)$$
Common causal explanation

- Local, non-conspiratorial joint common causal explanation: a partition \( \{ C_k \} \) in \( \Sigma \)

\[
p(A_m B_n | a_m b_n C_k) = p(A_m | a_m b_n C_k) p(B_n | a_m b_n C_k) \quad \text{(screening-off)}
\]

\[
p(A_m | a_m b_n C_k) = p(A_m | a_m C_k) \quad \text{(locality)}
\]

\[
p(B_n | a_m b_n C_k) = p(B_n | b_n C_k) \quad \text{(locality)}
\]

\[
p(a_m b_n C_k) = p(a_m b_n) p(C_k) \quad \text{(no-conspiracy)}
\]

- \( \{ C_k \} \) can be localized in any of the three different pasts
Motivation by Markov condition

Markov condition $\implies$ screening-off, locality and no-conspiracy
Clauser–Horne inequality

Joint CCS

Locality $\implies$ CH inequality

No-conspiracy

$$-1 \leq p(A_1 B_1 | a_1 b_1) + p(A_1 B_2 | a_1 b_2) + p(A_2 B_1 | a_2 b_1)$$

$$-p(A_2 B_2 | a_2 b_2) - p(A_1 | a_1) - p(B_1 | b_1) \leq 0$$

(which is equivalent to the CHSH inequality.)
EPR correlations

Conditional probabilities:
\[ p(A_m | a_m), \ p(B_n | b_n), \ p(A_m B_n | a_m b_n) \quad (m, n = 1, 2) \]

CH inequality is violated.

Therefore: no common causal explanation for EPR.
"Bell inequalities are relations between conditional probabilities valid under the locality assumption."
(Gisin, 2009)
Quantum ontology

- **Event space**: von Neumann lattice
- **Events**: projections
- **Probability**: quantum state
- **CH inequality**:

\[-1 \leq \phi (A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) \leq 0\]
Quantum ontology

- **Event space:** von Neumann lattice
- **Events:** projections
- **Probability:** quantum state
- **CH inequality:**

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*Then let us take the quantum ontology seriously!*

- **Common cause:**
  1. What is that?
  2. How it relates to the CH inequality?
Non-classical common cause system

- Non-classical probability space: \((\mathcal{P}(\mathcal{N}), \phi)\)

- Correlation: \(A, B \in \mathcal{P}(\mathcal{N})\)
  \[\phi(AB) \neq \phi(A)\phi(B)\]

- (Non-classical) CCS: partition \(\{C_k\}_{k \in K}\) in \(\mathcal{P}(\mathcal{N})\)
  \[
  \frac{\phi(C_kABC_k)}{\phi(C_k)} = \frac{\phi(C_kAC_k)}{\phi(C_k)} \cdot \frac{\phi(C_kBC_k)}{\phi(C_k)}
  \]
Non-classical common cause system

- **Non-classical probability space:** \((\mathcal{P}(\mathcal{N}), \phi)\)

- **Correlation:** \(A, B \in \mathcal{P}(\mathcal{N})\)

\[
\phi(AB) \neq \phi(A)\phi(B)
\]

- **(Non-classical) CCS:** partition \(\{C_k\}_{k \in K}\) in \(\mathcal{P}(\mathcal{N})\)

\[
\frac{\phi(C_kABC_k)}{\phi(C_k)} = \frac{\phi(C_kAC_k)}{\phi(C_k)} \times \frac{\phi(C_kBC_k)}{\phi(C_k)}
\]

- **Commuting / Noncommuting CCS:** \(\{C_k\}_{k \in K}\) is commuting / not commuting with \(A\) and \(B\)

- **Nontrivial CCS:** \(C_k \not\leq A, A^\perp, B\) or \(B^\perp\) for some \(k \in K\)
Joint common cause system

- Set of correlations: $A_m, B_n \in \mathcal{P}(\mathcal{N})$

$$\phi(A_m B_n) \neq \phi(A_m) \phi(B_n)$$

- Joint CCS: partition $\{C_k\}_{k \in K}$ in $\mathcal{P}(\mathcal{N})$

$$\frac{\phi(C_k A_m B_n C_k)}{\phi(C_k)} = \frac{\phi(C_k A_m C_k)}{\phi(C_k)} \cdot \frac{\phi(C_k B_n C_k)}{\phi(C_k)}$$

- A subtle point:

Joint CCS = local, non-conspiratorial joint CCS
Clauser–Horne inequality

Commuting joint CCS

(Locality) $\implies$ CH inequality
(No-conspiracy)

$-1 \leq \phi(A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 - A_1 - B_1) \leq 0$

EPR has no *commutative* common causal explanation
But why to demand commutativity between a cause and its effects?

Standard QM: operators *do not commute* with their time translates:

Harmonic oscillator: \( x(t) \equiv U(t)^{-1} x U(t) \)

\[
\begin{align*}
\left[ x(t), x \right] \psi_0 &= -\frac{i\hbar}{m\omega} \sin(\hbar \omega t) \psi_0 \neq 0
\end{align*}
\]
Noncommuting common causes

Noncommuting joint CCS

(Locality) \( \not\Rightarrow \) CH inequality
(No-conspiracy)

**Question:** Can a set of correlations violating the CH inequality have a noncommuting *joint* common causal explanation in AQFT?
An easier question: Can one correlation have a common causal explanation in AQFT? (Rédei 1997)

**Common Cause Principle (CCP):** If there is a correlation between two events and there is no direct causal (or logical) connection between them, then there always exists a common cause of the correlation.
Algebraic quantum field theory

- Poincaré covariant AQFT:

- Quantum Ising model:
Weak (Commutative/Noncommutative) CCP
(Commutative/Noncommutative) CCP
Common Cause Principles in AQFT

- Strong (Commutative/Noncommutative) CCP
**Proposition:** The Weak Commutative CCP *holds* in Poincaré covariant AQFT (Rédei, Summers, 2002).

**Question:** What about other AQFTs?
**Proposition:** The Weak Commutative CCP does not hold in quantum Ising model (Hofer-Szabó, Vecseryés, 2012a).

**Question:** What about abandoning commutativity?
**Proposition:** The Weak Noncommutative CCP holds in quantum Ising model (Hofer-Szabó, Vecsernyés, 2012b).
**Original question:** Can a set of correlations violating the CH inequality have a noncommuting *joint* common causal explanation in AQFT?
Correlations violating CH

- \( A_m = A(\vec{a}^m) \), \( B_n = B(\vec{b}^n) \): four projections \((m, n = 1, 2)\)
- \( \vec{a}^m, \vec{b}^n \): Bell directions
- \( \rho^s \): singlet state
- Maximal violation of the CH (CHSH) inequality.
Noncommuting common causes

... after some calculation ...

\[ \rho_C = 1 + \lambda U_{-\frac{1}{2}} U_{\frac{1}{2}} \]
\[ + \frac{1 + \lambda}{2} c_1 (U_{-\frac{1}{2}} + U_{\frac{1}{2}}) + \frac{1 - \lambda}{2} c_1' (U_{-\frac{1}{2}} - U_{\frac{1}{2}}) \]
\[ + \frac{1 + \lambda}{2} c_2 (U_0 - U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) - \lambda c_2 (U_{-1} U_0 U_1 + U_{-1} U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}} U_1) \]
\[ + \frac{1 - \lambda}{2} c_2' (U_0 + U_{-\frac{1}{2}} U_0 U_{\frac{1}{2}}) \]
\[ + \frac{1 + \lambda}{2} c_3 i (U_{-\frac{1}{2}} U_0 - U_0 U_{\frac{1}{2}}) + \frac{1 - \lambda}{2} c_3' i (U_{-\frac{1}{2}} U_0 + U_0 U_{\frac{1}{2}}) \]
\[ + \lambda c_1 c_2 (U_{-1} U_{-\frac{1}{2}} U_0 U_1 + U_{-1} U_0 U_{\frac{1}{2}} U_1) \]
\[ + \lambda c_2^2 (-U_{-1} U_1 + U_{-1} U_{-\frac{1}{2}} U_{\frac{1}{2}} U_1) \]
\[ + \lambda c_2 c_3 i (U_{-1} U_{-\frac{1}{2}} U_1 - U_{-1} U_{\frac{1}{2}} U_1). \]

**Answer:** Yes.
Proposition: (Hofer-Szabó, Vecsernyés, 2012c)
There is a *noncommuting* common cause \{C, C^\perp\} of the correlations \{(A_m, B_n)\}; and it can be localized in the shaded region.
**Conclusion**

**Classical case:** Common cause $\implies$ Bell inequality

**Quantum case:** Bell inequality
Conclusion

Classical case:  Common cause $\implies$ Bell inequality

Quantum case:  Common cause $\nRightarrow$ Bell inequality

The violation of the Bell inequality in AQFT does not exclude a set of correlations to have a joint common causal explanation if commutativity is abandoned.
In the noncommutative case the theorem of total probability does *not* hold. (No 'Hempelian' explanation.)

Are the (Strong/Weak) Noncommutative Joint CCPs valid in AQFT?

What are the ontological consequences of applying noncommutative common causes?
References


- V. F. Müller and P. Vecseryénő, ”The phase structure of $G$-spin models”, *to be published*


Quantum Ising model

Minimal double cones: $O_i^m$
Double cones: $O_{i,j}$, smallest double cone containing $O^m_i$ and $O^m_j$
Quantum Ising model

Net: $\mathcal{K}^m$, by integer time translation
‘One-point’ algebras

- Linear basis: $1, U_0$
- Minimal projections: $P = \frac{1}{2} (1 \pm U_0)$
- Commutation relations:

$$U_i U_j = \begin{cases} 
-U_j U_i, & \text{if } |i - j| = \frac{1}{2} \\
U_j U_i, & \text{otherwise}
\end{cases}$$
‘Two-point’ algebras

- **Linear basis:** $1, U_0, U_{1/2}, iU_0U_{1/2}$

- **Minimal projections:** $P = \frac{1}{2} \left( 1 + \vec{n} \cdot U \right), \quad \vec{n} \in \mathbb{R}^3$
**Dynamics**: automorphisms of $\mathcal{A}$ (Müller, Vecsernyés 2012)

Local primitive causality holds.
**Dynamics**: automorphisms of $\mathcal{A}$ (Müller, Vecsernyés 2012)

- Local primitive causality holds.
Correlations violating CH

- $A_m = A(\vec{a}^m)$, $B_n = B(\vec{b}^n)$: four projections ($m, n = 1, 2$)

- $\rho^s$: singlet state
Correlations violating CH

Directions:

maximally violating of the CH inequality ...
... or, equivalently, the CHSH inequality:

\[ \left| \phi(U_1(V_1 + V_2) + U_2(V_1 - V_2)) \right| \leq 2 \]

where

\[ U_m := 2A_m - 1 \]
\[ V_n := 2B_n - 1 \]
**Question:** Can these four correlations have a noncommutative joint common causal explanation?
Localization of the common cause

Weak joint common cause system

A_m

B_n

C_k
Localization of the common cause

Joint common cause system
Localization of the common cause

Strong joint common cause system
Localization of the common cause

- Weak joint common cause system: one needs only local primitivity and isotony (no dynamics)
Localization of the common cause

(Strong) joint common cause system: one needs also dynamics
Bell inequality in AQFT

- \( \mathcal{A} \) and \( \mathcal{B} \): two mutually commuting \( C^* \)-subalgebras of \( C \)
- **Bell operator** for \((\mathcal{A}, \mathcal{B})\): \( R \), an element of the set

\[
\mathbb{B}(\mathcal{A}, \mathcal{B}) \equiv \left\{ \frac{1}{2}(A_1(B_1 + B_2) + A_1(B_1 - B_2)) \middle| A_i = A_i^* \in \mathcal{A}; B_i = B_i^* \in \mathcal{B}; -1 \leq A_i, B_i \leq 1 \right\}
\]
Bell inequality in AQFT

- **Bell correlation coefficient** of a state $\phi$:

$$\beta(\phi, A, B) \equiv \sup \left\{ |\phi(R)| \mid R \in \mathbb{B}(A, B) \right\}$$

- The **Bell inequality** is violated if

$$|\beta(\phi, A, B)| > 1$$
**Proposition:** If $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras then there are some states violating the Bell inequality for $\mathcal{A} \otimes \mathcal{B}$ iff both $\mathcal{A}$ and $\mathcal{B}$ are non-abelian (Bacciagaluppi, 1994).

- Going over to von Neumann algebras ... (Landau 1987)
- Adding further constraints ... (Summer-Werner, 1988; Halvorson, Clifton, 2000)
- The above theorems apply in "typical" AQFTs ...
Joint common cause system

Joint CCS = local, non-conspiratorial joint CCS

Proof:

- Rewriting both the classical and the non-classical local, non-conspiratorial joint CCS in an *indexical form*.

- ’Translating’ quantum probabilities into classical conditional probabilities by the *Kolmogorovian Censorship Hypothesis*. 
Non-classical joint common cause system

Correlation:

\[ \phi(A_m B_n) \neq \phi(A_m) \phi(B_n) \]

Indexical notation:

\[ \phi_{C_k}(X) := \frac{\phi \circ E_c(XC_k)}{\phi(C_k)} = \frac{\phi(C_k X C_k)}{\phi(C_k)}. \]

Non-classical, local, non-conspiratorial joint CCS:

\[ \begin{align*}
\phi_{C_k}(A_m B_n) &= \phi_{C_k}(A_m) \phi_{C_k}(B_n) \\
\phi_{C_k}(A_m) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m B_n^\perp) \\
\phi_{C_k}(B_n) &= \phi_{C_k}(A_m B_n) + \phi_{C_k}(A_m^\perp B_n) \\
\phi_{C_k}(1) &= 1.
\end{align*} \]
Kolmogorovian Censorship Hypothesis

Let \((\mathcal{N}, \mathcal{P}(\mathcal{N}), \phi)\) be a non-classical probability space. Let \(\Gamma\) be a countable set of non-commuting selfadjoint operators in \(\mathcal{N}\). For every \(Q \in \Gamma\), let \(\mathcal{P}(Q)\) be a maximal Abelian sublattice of \(\mathcal{P}(\mathcal{N})\) containing all the spectral projections of \(Q\). Finally, let a map \(p_0 : \Gamma \to [0, 1]\) be such that

\[
\sum_{Q \in \Gamma} p_0(Q) = 1, \quad p_0(Q) > 0.
\]

Then there exists a classical probability space \((\Omega, \Sigma, p)\) such that for every projection \(X^Q\) in any \(\mathcal{P}(Q)\) there exist events \(X^Q_{cl}\) and \(x^Q_{cl}\) in \(\Sigma\) such that

\[
X^Q_{cl} \subset x^Q_{cl}
\]

\[
x^Q_{cl} \cap x^R_{cl} = 0, \quad \text{if} \ Q \neq R
\]

\[
p(x^Q_{cl}) = p_0(Q)
\]

\[
\phi(X^Q) = p(X^Q_{cl} | x^Q_{cl})
\]
Classical joint common cause system

Correlation:

\[ p(A_m \land B_n \mid a_m \land b_n) \neq p(A_m \mid a_m) \cdot p(B_n \mid b_n) \]

Indexical notation:

\[ p_{C_k}(X \mid x) := \frac{p(X \land C_k \mid x)}{p(C_k)}. \]

Classical, local, non-conspiratorial joint CCS:

\[
\begin{align*}
p_{C_k}(A_m \land B_n \mid a_m \land b_n) &= p_{C_k}(A_m \mid a_m \land b_n) \cdot p_{C_k}(B_n \mid a_m \land b_n), \\
p_{C_k}(A_m \mid a_m \land b_n) &= p_{C_k}(A_m \mid a_m \land b_n'), \\
p_{C_k}(B_n \mid a_m \land b_n) &= p_{C_k}(B_n \mid a_m' \land b_n), \\
p_{C_k}(\Omega \mid a_m \land b_n) &= 1.
\end{align*}
\]
Quantum Ising model

- **Cauchy surface net**: $K_{CS}^m$, poset of double cones based on the Cauchy surface
‘One-point’ algebras

- **Linear basis**: $1, U_0$
- **Minimal projections**: $P = \frac{1}{2} (1 \pm U_0)$
- **Commutation relations**:

$$U_i U_j = \begin{cases} 
  -U_j U_i, & \text{if } |i - j| = \frac{1}{2} \\
  U_j U_i, & \text{otherwise}
\end{cases}$$
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- **Linear basis:** 1, $U_0$

- **Minimal projections:** $P = \frac{1}{2} (1 \pm U_0)$

- **Commutation relations:**

\[
U_i U_j = \begin{cases} 
-U_j U_i, & \text{if } |i - j| = \frac{1}{2} \\
U_j U_i, & \text{otherwise}
\end{cases}
\]
‘Three-point’ algebras

- **Linear basis:**
  \[ 1, U_{-\frac{1}{2}}, U_0, U_{\frac{1}{2}}, iU_{-\frac{1}{2}}U_0, iU_0U_{\frac{1}{2}}, U_{-\frac{1}{2}}U_{\frac{1}{2}}, U_{-\frac{1}{2}}U_0U_{\frac{1}{2}} \]

- **Minimal projections:**
  \[ P = P(\vec{n}), \quad \vec{n} \in \mathbb{R}^3 \]

- **Two dimensional projections:**
  \[ P = P(\vec{n}, \vec{n}'), \quad \vec{n}, \vec{n}' \in \mathbb{R}^3 \]