Imre Lakatos's Philosophy of Mathematics

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Introduction

Imre Lakatos (1922-1974) is known world-wide by at least two different groups of philosophers. For the philosophers of science, he was a great defender of scientific rationality: he gave up the idea of 'instant' methodological rationality and substituted it with the historiographical methodology based on the notion of 'rational reconstruction'. For the philosophers of mathematics, he was a great attacker at the formalist school, and instead of the formal-axiomatic features of mathematics he put the emphasis on heuristics, the historical processes of theory building and conceptualisation. These two 'incarnations' of Lakatos are rarely discussed within a unique philosophical framework, and not without reason: Lakatos himself never really tried to harmonise these two fields of interest. There are signs showing that he intended to build a common ground for them, but perhaps this project could not even take a very definite shape before his early death.

In this paper I shall concentrate on only one of these topics: his philosophy of mathematics; and my primary aim is to give a brief and coherent summary of his ideas concerning this field. It should be noted, however, that his philosophy of mathematics and that of science obviously have some characteristics in common, since they were contrived by the same man, and not at a great temporal distance in his academic life, when he was under very similar intellectual influences. My secondary purpose therefore is to build this summary in a way that is in accordance with his general philosophical theory of science, that is, to give a mild 'rational reconstruction' of his ideas.[1]

The context of Lakatos's philosophy of mathematics

In the Acknowledgements of his doctoral thesis wrote on the philosophy of mathematics, Lakatos mentions three authors as the most influential sources for his philosophy. This very interesting selection puts his philosophy into a context that helps us understand his motives and his purposes. The sources he lists are the following:

1. *Popper's falsificationist theory*. At the end of the 1950s and the beginning of the 1960s, when he worked mainly on the philosophy of mathematics, Lakatos was

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[1] For further comparison of the two fields, the reader is suggested to consult the reference list given below.

under the very strong influence of Karl Popper.[2, 3] Popper used the term 'falsification' for the refutation of theories by empirical facts, and his main thesis was that there can be no empirical verification of scientific theories. Lakatos applied this emphasis on refutations on a discipline that is known to be based exclusively on proofs, and showed that when mathematicians do mathematics then they very often *refute* one another's theories. This emphasis will take us from the ideal picture of mathematics (which works only with proofs) to the actual mathematical activity.

2. *George Pólya's heuristics.* Lakatos was introduced to Pólya's ideas when he was still in Hungary and worked for the Mathematical Institution.[4] Pólya emphasised the role of *heuristic rules* in the actual practice of mathematics: his concern was *informal* mathematics, which deals with the way how mathematicians reach their results. Lakatos sided with this hardly existent standpoint in the philosophy of mathematics, and used these ideas in his assault on the formalist school.

3. *Hegel's dialectics.* The Hegelian influence came very early for Lakatos, in his university years in Hungary, when he became devotedly involved in the Marxist-Lukacsian philosophy and ideology.[5] However, the trace of the Hegelian ideas can be seen also in his latest philosophical writings, and it is particularly present in his philosophy of mathematics. In his philosophical education, Lakatos learnt that the investigation of a certain subject cannot be limited to a study in the fixed conceptual frame of a rigid system, but it should turn to the *conceptual dialectics* that helps the subject develop and show itself. He learnt that the stress is on progress, and not on static states; the philosophy of mathematics must be a philosophy of *development*, not of static certainty – less of knowledge and more of the growth of knowledge.

Keeping these different influences in mind, let us see how and where Lakatos tried to place his own philosophy of mathematics.

The basic types of deductive sciences

In order to see where mathematics is located in the methodological map of all sciences, Lakatos gave a classification of the ('well-working') sciences, i.e. the axiomatic-deductive systems. Forgetting about the subtleties of the classification, I would like to distinguish between two basic types, or the two 'poles', of these systems: there are *Euclidean* systems, and on the other hand there are *quasi-empirical* systems. (See Fig. 1) What is shared by both of these kinds of systems is the axiomatic-deductive 'spirit'; that is, both take some statements as basic ('axioms') and try to derive by means of logical deduction further statements ('theorems') from

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[2] Not much later, he formulated his theory in the philosophy of science as an 'extension' of Popper's philosophy.
[3] At least two titles of his mathematical writings refer to Popper's (most important) works: *Logic of Mathematical Discovery* rhymes with Popper's *Logic of Scientific Discovery*, while *Proofs and Refutations* consonances with Popper's *Conjectures and Refutations*. – See the titles explained later.
[4] He even translated Pólya's *How to Solve It* into Hungarian.
[5] It seems that first he did not meet Hegel's ideas in their original form, but became familiar with them only through Lukacs's interpretation in his *History and Class-Consciousness*.

these axioms. The main difference lies in the way in which these systems are connected to 'truth' and 'falsity' – why we assign to them epistemological role and credibility.
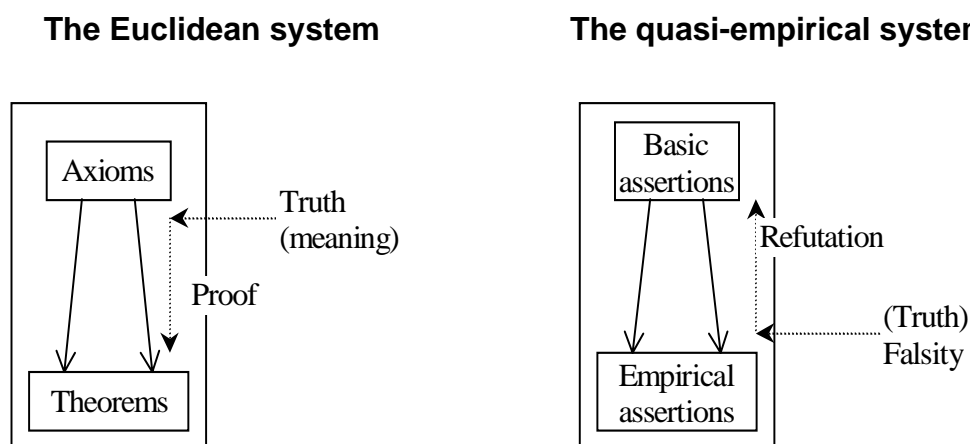
The Euclidean system **The quasi-empirical system**

Axioms

Truth
(meaning)

Proof

Theorems

Basic
assertions

Refutation

(Truth)
Falsity

Empirical
assertions

Figure 1

In the case of the Euclidean sciences, truth is 'injected' to the system at the 'top', at the level of axioms. From here truth is inherited 'downward' to the theorems automatically, since it belongs to the nature of (valid) logical deduction that the truth of the premises ensures the truth of the conclusion. This strict, infallible inheritance of truth along the 'truth preserving channels' of deduction is called *proof*. Provided that the truth of the axioms is given somehow, the truth and validity of the whole system is beyond any doubt. It's worth noting that the deductive channels are not only 'truth preserving' but 'meaning preserving' as well, so meaning of the terms also gets into the system at the level of axioms (with definitions), and is inherited along the inferences.

In the case of quasi-empirical systems, the situation is quite the opposite: the source of truth is the comparison with experience (understood in a broad sense), therefore it is the final, 'bottom' statements to which one can directly assign truth values. But deductive inference is such that truth cannot be inherited in the 'reverse' direction or 'upward', for the truth of the conclusion does not enable one to decide whether or not the premises are true. (We might as well say that induction is not a legitimate procedure in this general deductive scheme.[6]) Falsity, on the other hand, is inherited upward, since the falsity of the conclusion entails the falsity of (at least one of) the premises. Quasi-empirical systems therefore work with *refutation*.[7] We mention that meaning cannot be inherited 'upward' either, so in the case of empirical sciences there is a notable freedom about the meanings of terms, unlike with Euclidean sciences. Later this point will prove to be essential for mathematics, after we learnt that mathematics, surprisingly, belongs to this second group.

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[6] Lakatos discusses the case of the logic of probability as a candidate for exception, but he excludes this case by suggesting that for the cognitive certainty chased by science, 'probable' truths cannot be regarded as satisfactory a solution.

[7] See the previous section about Popper's influence on Lakatos with his devotion to refutations.

For most part of the history of science, mathematics was regarded not only as 'one' of the Euclidean sciences, but as the archetype for all of them – as the stronghold of rationality from where we can start our final assault on those uncertain parts of human knowledge which still resist the power of pure (Euclidean) reason. But here comes the great epistemological difficulty: in order to place mathematics on the throne of all Euclidean sciences, we have to be able to give a credible account of the mechanism that *makes the axioms true*. Lakatos discusses (mainly) two candidates for the epistemology of Euclidean mathematics: *logicism* and *formalism*.

The failure of Euclidean epistemologies

According to the logicist philosophers (primarily Gottlob Frege and the young Bertrand Russell), the evidence for the axioms of mathematics arises from *logical intuition*. In other words, the axioms are logically true statements (or theorems of logic), and their negations are self-contradictions. In this case, infallibility of the axioms is given on the same basis as the validity of deduction (i.e. the power of logic), and since logic has some intimate relation to the nature of thinking, mathematics is therefore unquestionably true.

But the failure of the logicist programme became obvious soon after its birth, and ironically it was one of its partisans, Russell, who put his finger on the incurable disease (the Russell-paradox) in Frege's formal system. "Logical intuition", from which they intended to derive the validity of mathematics, and which is given its theoretical shape by naïve set theory, is inherently inconsistent. While consistency can be reached by substituting naïve set theory with axiomatic set theory, this new formal shape will be far from representing any infallible logical intuition[8], so it means the betrayal of the logicist programme.

The next (and, according to Lakatos, hopefully the last) attempt at establishing a Euclidean framework for mathematics was David Hilbert's formalist programme. If the failure of the logicist programme was due to inherent inconsistencies, the main task is to ensure consistency for mathematics. The formalist philosopher therefore identifies mathematical theories with purely formal syntactical systems (calculi) that have to satisfy two main conditions: (i) the calculus must be consistent[9], and (ii) it must be complete with respect to negation, that is, one must be able to either prove or refute every theorem composed in it (found true or false by 'intuitive', less formal methods).

Lakatos dates the failure of this programme to 1931, when Kurt Gödel's two incompleteness theorems proved that both of the formalist programme's criteria are untenable: there can be no formal theory (strong enough for mathematical purposes) which is syntactically complete; and none of these theories can prove its own inconsistency (so there is no provably inconsistent basic theory to which the

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[8] Some axioms of the Zermelo-Fraenkel set theory (or any other axiomatics of set theory) are far from being 'obviously true', and their validity (like in the case of the axiom of choice) was debated for decades in the first half of the century.

[9] One could say that consistency substitutes the requirement of truth – so the formalist programme is *not* a purely Euclidean epistemology for mathematics, but it is in the sense that its main methodological notion is *proof*.

consistency of all other theories could be reduced). Lakatos admits that this failure is not acknowledged yet by most of mathematicians, and they try to cure the disease in metalogics, proof theory and other fields; but he maintains that every attempt of this kind can be shown to transcend the original (finitist) limits of Hilbert's programme.[10] Intuitions about consistency and completeness are inadequate to substantiate a Euclidean philosophy of mathematics.

Instead of discussing further and less important attempts and failures, Lakatos draws the historically based conclusion (which is not logically strict, but is demonstrated by the preceding considerations) that mathematics is *not* a Euclidean science. This means, on the one hand, that mathematics is not infallible: we were not able to find a mechanism that ensures the necessary truth of the axioms. It also means, on the other hand, the mathematics is not purely demonstrative: the axioms cannot carry the required authority that would make the method of proof credible – indeed, we often look for the suitable axioms in order to ensure the validity of certain theorems we want to prove. (Modifying or abandoning the basic statements is *refutation*, the epistemological opposite of proof.) Finally, mathematics cannot be purely formal, since formal systems cannot live up to the basic expectations we want to raise for them.

The marriage between philosophy and history of mathematics

After this brief historical account we can see that attempts at *prescribing* for mathematics what it should be like, or how it should work in the ideal case, turned out to be illusory. Philosophy of mathematics has to start from *describing* mathematics as it is. For this, philosophy has to "wed" the history of mathematics: it has to look at the real events to find there the method of progress, the method which succeeds in bringing about the growth of mathematical knowledge[11]. This idea is expressed by the famous thesis: "The history of mathematics, lacking the guidance of philosophy, [is] blind, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics, is empty."[12]

The second part of the above statement can be easily explained by what we have already seen: every philosophy that does not consider the history of its own subject fails to grasp anything, since it will not understand the basic working mechanism of the subject field.[13] However, the first part of the thesis is much more problematic, since not many historians will admit that historical research cannot proceed without using a philosophical framework. But Lakatos does not acknowledge the existence of philosophically unbiased historiographies: according to his view, when historians of science turn to the actual historical events, they already have a preconception of what

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[10] One example is the Gentzen-type consistency proof for arithmetic based on transfinite induction.

[11] See above Hegel's influential role in Lakatos's thinking.

[12] This is actually a paraphrase of Immanuel Kant's thesis on the relation between intuition and intellect. Lakatos expressed the same view for the case of science (and not of mathematics), and he put this as the starting sentence of one of his most important papers on the philosophy of science ("History and its Rational Reconstructions").

[13] In the same years when Lakatos expressed this view, a very similar claim was formed by Thomas Kuhn, historian of science, and today there is hardly any philosopher of science who will not grant the validity of this requirement.

constitutes the nature of the given discipline. The set of historical data is too large and confused to be handled, with no intrinsic system or method in them, so we need to have certain preconceptions and biases: historians unavoidably give *rational reconstructions* of the actual history. Granted this, instead of trying to get rid of preconceptions (which is impossible) we must consciously dwell on this point and, with our historical research, we should aim the question: 'What is (this) science?'[14]. If philosophy is inevitably present in the history of science, then we should make use of it.[15]

What we first see, having a look at the history of mathematics after this analysis of the formalist school's failure, is that mathematical theories are not given to us as formal calculi. Instead, formal calculi are built by mathematicians in order to strictly conceptualise *informal theories*. There is always an informal theory prior to the formal system, and the nature of this vague theory is cleared up exactly by fixing the formal frame. Moreover, this formalisation process is the very mechanism of mathematical discovery: the growth of mathematical knowledge is achieved by building formal accounts of the informal intuition. Concerning epistemology again, the truth of mathematical theories is therefore reduced to the truth of these informal theories, but this second question is not answered by Lakatos in one unique traditional way. Abstraction from experience; intellectual view of the realm of eternal Platonic entities; the intuition given by the conceptual construction of objects – all these accounts can be used (changing from case to case) when assigning truth to the informal theory captured. But the philosophy of mathematics is *not* concerned with this question, but rather with the way in which this fixed system of formal concepts is established, with the process that leads from conjectures to knowledge. This philosophy is not epistemology but methodology, or *heuristics*.

The method of proofs and refutations

Lakatos's most important writing on the philosophy of mathematics is *Proofs and Refutations*, a series of articles (later published as a book) that he extracted from his (second) doctoral dissertation entitled *Essays on the Logic of Mathematical Discovery* (Cambridge, 1961). The main part of this work is a case study, a 'rational reconstruction' of a historical process which, according to the author, is excellently suitable for illustrating the growth of mathematical knowledge. This process is the 18th-20th century development of polyhedrons. The 'internal history' (or rational reconstruction) is supplemented with an 'external history' (the series of real historical events) given in footnotes.[16] The main steps of the general process are the following (see Fig. 2):

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[14] Or even: 'What is (scientific) rationality?'

[15] The concept of "rational reconstruction" brought about lively debates, and even today it is very often scorned and persecuted among historians. The general discussion of this notion would lead us out of the field of history of mathematics, and would require a detailed analysis of Lakatos's philosophy of science.

[16] In "History and its Rational Reconstructions" Lakatos expresses his idea for all historiographies of science that it is a good way to write history when the reconstruction composes the main text, while the real history is given in footnotes. This is the only case (anticipating his later idea) when he follows his own advice.
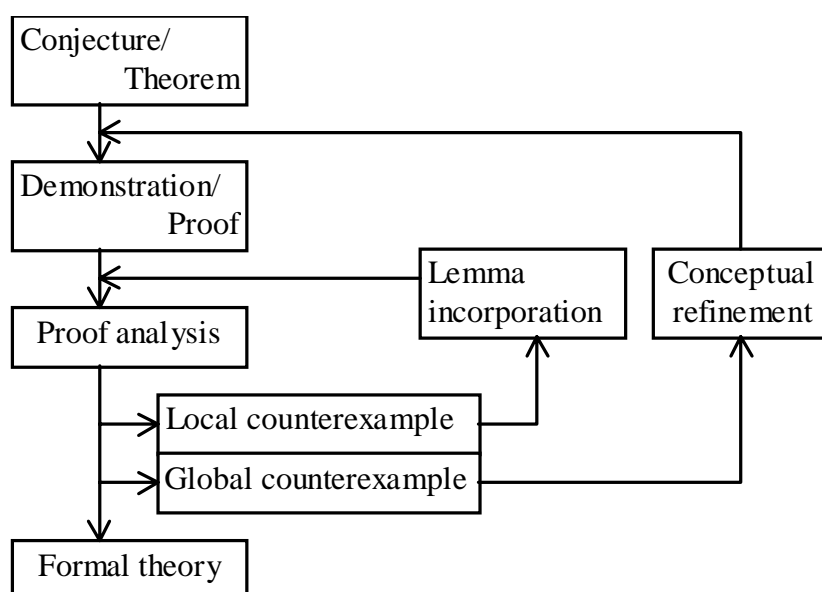
Heuristics in *Proofs and Refutations*

Figure 2

1. *Naïve conjecture*. Mathematical research always starts from a problem (and it should be noted that always ends in a problem too). The problem is the intuitive recognition of some regularity or connection that cannot be expressed within the well-defined, formal bounds of existing theories. Therefore, the expression of the naïve conjecture does not entail the possibility of proof: we need a formal theory in which the statement can become a proven theorem. (The sources of this 'recognition' can be various and contingent, and they are not interesting for the philosophy of mathematics – as we observed when we dispensed with epistemology.) In the examined case, this recognised regularity is the *Descartes-Euler conjecture*, i.e. for every polyhedron, the number of vertices plus the number of faces minus the number of edges equals 2.

2. *Proof analysis*. The purpose now is to build a formal theory in which the naïve conjecture can be expressed precisely and proved by means of deduction. First we give a 'naïve' proof for our statement, an 'intuitive' demonstration of its general validity. In our case this demonstration is Cauchy's argument: seeing the polyhedrons as rubber sheets, if we imagine that we cut them along one of the edges and stretch them flat, we can 'see' that the statement is true. But the community of mathematicians will abuse the looseness of this 'rubber sheet' notion of polyhedrons, and they will easily come up with counterexamples: cases for which the conjecture does not hold. All *criticisms* and counter-criticisms become vital now, they are the engines of the evolution of theories.[17]

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[17] The role of criticisms was emphasised by Karl Popper as being essential to the rationalist attitude. For Lakatos, who does not admit the existence of any fixed scientific method of rationality, this role of criticisms and the rationalist attitude become even more indispensable.

Counter-examples can be divided into two main groups: local and global counter-examples.[18] Local ones are those that do not refute our 'theorem' in general, they only refute one of the 'hidden lemmas' we unconsciously included in the formulation of our theorem. In this case we substitute the 'guilty lemma' with another one which now excludes the validity of the counter-example. Global counter-examples, on the other hand, are the ones that do refute our initial theorem. However, we do not throw away our theorem and its proof, but we *modify* the concepts and notions involved, or make it more precise what the statement is about (e.g. what polyhedrons are).

In this way, through the dialectical interaction of proofs, refutations and proof analyses, a formal system of precise concepts is gradually created, and this will shape a new formal mathematical theory.[19]

3. *Deductive theory*. Finally, the 'research programme' ends in the new formal theory. All the meanings of terms are fixed within the axiomatic system, and many theorems (probably including the original one) can be deduced. (In our case study, the new theory is the axiomatic system of algebraic topology – see *Proofs and Refutations*.) In this last stage, the mathematical activity is reduced to 'puzzle solving'[20] (which theorems can be proven in the theory), and no exciting problem will emerge. For the formailst philosophy of mathematics, this is the only interesting, real form of mathematical activity – for Lakatos, this is the most boring and least interesting one.

It is worth noting that, according to Lakatos, the same scheme of development can be shown in the case of ancient Greek mathematics as well, for the evolution of geometry from Thales to Euclid.[21] And, more even interestingly, Lakatos argues for the validity of this scheme for the birth of modern physics: from Kepler's 'naïve conjectures' (which were not provable within any existing physical theory at the time of their recognition) to Newton's axiomatic foundation of mechanics. There are some signs, as we mentioned in the Introduction, that, in the last few years of his life, Lakatos did not see mathematics and natural sciences different, neither in methodology nor in subject – but, unfortunately, for the explicit formulation of these views he had not enough time left before his death.

Turning back to mathematics, we have to mention that this example shown in *Proofs and Refutations* illustrates more than the way in which new formal theories are born. It illustrates the very nature of mathematical progress: how proof analytical activity helps capture the intrinsic subject. Mathematics *is* based on proof in the end, when one field of interest reaches its final, axiomatic stage: so much should be granted for the formalist school – but then this field becomes empty and dead. The introduction of the historical dimension of mathematics serves the purpose of seeing mathematics as a *progress*: in philosophy, what we are interested in is not the formal features of

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[18] For illustrations of the following cases, see *Proofs and Refutations* itself.

[19] During the evolution of this 'research programme', the 'hard core' is determined and the 'positive heuristics' of the programme is activated – see Lakatos's terminology for the philosophy of science, which he partly applied to the philosophy of mathematics as well.

[20] This term is borrowed by Lakatos from Thomas Kuhn.

[21] He refers to his former teacher and friend, Árpád Szabó, the well-known historian of mathematics.

knowledge, but rather the growth of knowledge. As Lakatos emphasises: in mathematics, all growth in rigour is transformed to be a growth in content[22]; that is, every criticism that increases the strictness of methodology and terminology in one question, also increases the range of our knowledge and understanding available for scientific (i.e. deductive) methods.

Conclusion

It seems very likely that the originality and the deep insights in Lakatos's philosophy are partly due to his very diverse and varied training: from a Hegelian-Lukacsian philosophy to Cambridge's formal analytical way of argumentation, he received very different influences during his academic life. These formally contradicting views gave birth to very live and efficient ideas in his philosophy, though he never took the chance to clarify a final, complete philosophical system – something that he put aside as uninteresting in case of mathematical theories. It is rather the approach than the actual statements that should be valued highest among all that we inherited from his ideas: the emphasis on the historical dimension; the devotion to development and progress; and the attempt to see mathematics not as a formal thing that is alien to us but as it is worked by mathematicians. He brought down mathematics from its supposedly given divine perfection to the realm of human beings who are responsible for the achievement of an ever higher degree of its perfection – which is, as it seems to be, a Copernican revolution in the philosophy of mathematics.

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[22] He connects this process to the 'cunning of Reason' – an important notion for Hegel.